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PHASE TRANSITION IN EQUILIBRIUM FLUCTUATIONS OF SYMMETRIC SLOWED EXCLUSION

TERTULIANO FRANCO, PATRÍCIA GONÇALVES, AND ADRIANA NEUMANN

ABSTRACT. We analyze the equilibrium fluctuations of the density, current and tagged particle in symmetric exclusion with a slow bond. The system evolves in the one-dimensional lattice and the jump rate is everywhere equal to one except at the slow bond where it is $\alpha n^-\beta$, where $\alpha,\beta \geq 0$ and n is the scaling parameter. Depending on the regime of β, we find three different behaviors for the limiting fluctuations whose covariances are explicitly computed. In particular, for the critical value $\beta = 1$, starting a tagged particle near the slow bond, we obtain a family of gaussian processes indexed in α, interpolating a fractional brownian motion of Hurst exponent $1/4$ and the degenerate process equal to zero.

1. INTRODUCTION

The exclusion process is a standard interacting particle system, widely studied in Probability and Statistical Mechanics. Informally, such model corresponds to particles performing continuous time random walks in a lattice, except when a particle tries to jump to an already occupied site. In such case, the jump is forbidden and the particle has to wait a new random time.

There is an intensive research on the behavior of exclusion processes in many different aspects and from varied points of view. In particular, on the behavior of exclusion processes in random/non homogeneous medium, see for instance [2, 3, 4, 6].

In this paper we analyze the fluctuations of the one-dimension symmetric exclusion process with a slow bond, for which the hydrodynamic limit was treated in [4, 5]. The dynamics of this model can be described as follows. On the one-dimensional lattice, it is allowed at most one particle per site. To each bond is associated a Poisson clock. When this clock rings, the occupation variables at the vertices of the bond are interchanged with a certain rate. Of course, if both the sites are occupied or empty, nothing happens. All bonds have a Poisson clock of parameter one, except one special bond, the *slow bond*, in which the Poisson clock has parameter $\alpha n^{-\beta}$, where $\alpha,\beta \geq 0$ and n is the integer scaling parameter. At the end n is lead to infinity. The process starts from the equilibrium measure, namely a Bernoulli product measure of parameter $\rho \in (0,1)$, and it is seen in the diffusive time scale, or else, in times of order n^2.

We are concerned with the fluctuations, that is, the Central Limit Theorem (C.L.T.) for the density, the current of particles through a fixed bond and the tagged particle. Such results are well known for the classical symmetric exclusion, where all the Poisson clocks have parameter one. For the density, the fluctuations are given by a generalized Ornstein-Uhlenbeck process, while the fluctuations of the current and the tagged particle are both given by the fractional brownian motion of Hurst exponent $1/4$.

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The introduction of the slow bond changes dramatically the scenario. Not so intuitively, the value $\beta = 1$ is critical. For $\beta \in [0,1)$, we obtain *ipsis litteris* the same results for the fluctuations of the symmetric exclusion just mentioned. This means that, in this case, the jump rate at the slow bond is not sufficiently strong in order to change the macroscopic behavior of the system. Nevertheless, the proof of last result is not straightforward and requires a Local Replacement which is sharp for this regime of β. For $\beta \in (1,+\infty]$, it is proved here that the fluctuations of the density are driven by the semigroup of the heat equation with Neumann's boundary conditions. This means that for this regime of β, the slow bond splits the system into two separate regions in which the macroscopic dynamics evolves independently.

Finally, at the critical value $\beta = 1$, the generalized Ornstein-Uhlenbeck process obtained is driven by the semigroup of the partial differential equation

$$\begin{cases} \partial_t u(t,x) = \partial^2_{xx} u(t,x), & t \geq 0, x \in \mathbb{R}\backslash\{0\} \\ \partial_x u(t,0^+) = \partial_x u(t,0^-) = \alpha\{u(t,0^+) - u(t,0^-)\}, & t \geq 0 \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

if the slow bond is located near to origin. If the slow bond is located elsewhere, the result is the same, but with the boundary conditions stated above for the corresponding macroscopic point. We remark that last equation is similar to the heat equation with a boundary condition of Robin's type, but relating the positive and negative half-lines. Notice that, for this regime of β, the parameter α survives in the limit. We also mention that last result, for $\alpha = 1$, exhibits explicitly the Ornstein-Uhlenbeck process as obtained in [3], considering the measure W there as being the Lebesgue measure plus a delta of Dirac, but in infinite volume.

Provided by the density fluctuations, we obtain, for the three regimes of β, the corresponding current fluctuations and we compute explicitly the covariances for the limiting gaussian processes. It is of worth to remark the behavior of the fluctuations of the current through the slow bond. For $\beta \in [0,1)$ we get a fractional brownian motion of Hurst exponent $1/4$ and for $\beta \in (1,+\infty]$ we get the degenerate process equal to zero. For $\beta = 1$, the current fluctuations are given by a family of gaussian processes indexed in α interpolating the fractional brownian motion of Hurst exponent $1/4$ and the degenerate process equal to zero. By this, we mean that we can recover these two processes from the case $\beta = 1$ by taking the limit as $\alpha \to +\infty$ or as $\alpha \to 0$, respectively, being the convergence in the sense of finite dimensional distributions.

Lastly, as a consequence of the previous result, it is straightforward to obtain the Central Limit Theorem for a tagged particle. In this case, we consider as initial measure the Bernoulli product measure conditioned to have a particle at a given site. Therefore, the system is no longer in equilibrium, but anyhow we can use the previous result to deduce the behavior of a tagged particle in this non-equilibrium situation. Following [7, 10] and since we are in one dimension, the aforementioned result follows from relating the position of a tagged particle with the current and the density of particles.

The paper is divided as follows. In Section 2, we introduce notations and state the results. In Section 3, we present the C.L.T. for the density of particles. In Section 4, we get an explicit formula for the semigroup of (1). In Section 5 we give a martingale characterization of the generalized Ornstein-Uhlenbeck processes obtained in the fluctuations of the density of particles. In Section 6, we prove the Central Limit Theorem for the current. Section 7 contains some useful estimates that we will use along the text.

2. DEFINITIONS AND MAIN RESULTS

2.1. **The model.** The symmetric simple exclusion process with conductances $\xi^n_{x,x+1} \geq 0$ is a Markov Process $\{\eta_t : t \geq 0\}$, with configuration space $\Omega := \{0,1\}^{\mathbb{Z}}$. We denote by η the configurations of the state space Ω so that $\eta(x) = 0$, if the site x is vacant, and $\eta(x) = 1$, if the site x is occupied. Its infinitesimal generator $\mathscr{L}_n$ acts on local functions $f : \Omega \to \mathbb{R}$ as

$$(\mathscr{L}_n f)(\eta) = \sum_{x\in\mathbb{Z}} \xi^n_{x,x+1} \Big[f(\eta^{x,x+1}) - f(\eta) \Big], \tag{2}$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$:

$$(\eta^{x,x+1})(y) = \begin{cases} \eta(x+1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x+1, \\ \eta(y), & \text{otherwise.} \end{cases}$$

We define the symmetric exclusion with a slow bond by taking the conductances as

$$\xi^n_{x,x+1} = \begin{cases} \alpha n^{-\beta}, & \text{if } x = -1, \\ 1, & \text{otherwise}. \end{cases}$$

We notice that when $\beta = 0$ and $\alpha = 1$, the process becomes the well known symmetric simple exclusion process. We are interested in analyzing the behavior of the process when $\beta \in (0,+\infty]$.

A simple computation show that the Bernoulli product measures $\{\nu_\rho : 0 \leq \rho \leq 1\}$ are invariant, in fact reversible, for the symmetric simple exclusion process with conductances, in particular also for the considered process. More precisely, ν_ρ is a product measure over Ω with marginals given by $\nu_\rho\{\eta : \eta(x) = 1\} = \rho$, for x in $\mathbb{Z}$.

Denote by $\{\eta_{tn^2} : t \geq 0\}$ the Markov process on $\{0,1\}^{\mathbb{Z}}$ associated to the generator $n^2\mathscr{L}_n$. Let $\mathscr{D}(\mathbb{R}_+,\Omega)$ be the path space of càdlàg trajectories (continuous from the right with limits from the left) with values in Ω. For a measure μ_n on Ω, denote by $\mathbb{P}^\beta_{\mu_n}$ the probability measure on $\mathscr{D}(\mathbb{R}_+,\Omega)$ induced by the initial state μ_n and the Markov process $\{\eta_{tn^2} : t \geq 0\}$. Expectation with respect to $\mathbb{P}^\beta_{\mu_n}$ will be denoted by $\mathbb{E}^\beta_{\mu_n}$. To simplify notation, we will denote $\mathbb{P}^\beta_{\nu_\rho}$ by $\mathbb{P}^\beta_\rho$. We define also $\chi(\rho) := \rho(1-\rho)$, the so-called *static compressibility* of the system.

2.2. **The Operators Δ_β and ∇_β.** We introduce some spaces we will use in the sequel.

Definition 2.1. *Let $L^2_\beta(\mathbb{R})$ be the space of functions $H : \mathbb{R} \to \mathbb{R}$ with $\|H\|_{2,\beta} < \infty$, where*

$$\|H\|^2_{2,\beta} = \begin{cases} \int_{\mathbb{R}} (H(u))^2 du, & \text{if } \beta \neq 1 \\ \\ \int_{\mathbb{R}} (H(u))^2 du + (H(0))^2, & \text{if } \beta = 1. \end{cases}$$

Notice that, for $\beta \neq 1$, the norm $\|\cdot\|_{2,\beta}$ is the usual L^2-norm with respect to the Lebesgue measure that we denote by λ. For $\beta = 1$, the norm $\|\cdot\|_{2,\beta}$ is the L^2-norm with respect to the measure $\lambda + \delta_0$, where δ_u denotes the Dirac measure at the point $u \in \mathbb{R}$.

In the sequel we will denote

$$H(0^+) := \lim_{\substack{u\to 0,\\ u>0}} H(u) \quad \text{and} \quad H(0^-) := \lim_{\substack{u\to 0,\\ u<0}} H(u).$$

For $k \in \mathbb{N}$, we denote by $H^{(k)}(x)$, the k^{th}-derivative of a function $H : \mathbb{R} \to \mathbb{R}$ at the point $x \in \mathbb{R}$. For $k = 0$, $H^{(0)}(x)$ means $H(x)$.

Definition 2.2. *Define $\mathscr{S}(\mathbb{R}\backslash\{0\})$ as the space of functions $H \in C^\infty(\mathbb{R}\backslash\{0\})$ and continuous from the right at $x=0$, for which*

$$\|H\|_{k,\ell} := \sup_{x\in\mathbb{R}\backslash\{0\}} |(1+|x|^\ell)H^{(k)}(x)| < \infty,$$

for all integers $k,\ell \geq 0$, and $H^{(k)}(0^-) = H^{(k)}(0^+)$, for all k integer, $k \geq 1$.

Next, we present the domains for Δ_β and ∇_β.

Definition 2.3. *For $\beta \in [0,1)$, we define $\mathscr{S}_\beta(\mathbb{R})$ as the subset of $\mathscr{S}(\mathbb{R}\backslash\{0\})$ composed of functions H satisfying*

$$H(0^-) = H(0^+).$$

Notice that the space above is nothing more than the usual Schwartz space $\mathscr{S}(\mathbb{R})$. Fix now $\alpha > 0$.

Definition 2.4. *For $\beta = 1$, we define $\mathscr{S}_\beta(\mathbb{R})$ as the subset of $\mathscr{S}(\mathbb{R}\backslash\{0\})$ composed of functions H satisfying*

$$H^{(1)}(0^+) = H^{(1)}(0^+) = \alpha\{H(0^+) - H(0^-)\}.$$

Definition 2.5. *For $\beta \in (1,+\infty]$, we define $\mathscr{S}_\beta(\mathbb{R})$ as the subset of $\mathscr{S}(\mathbb{R}\backslash\{0\})$ composed of functions H satisfying*

$$H^{(1)}(0^+) = H^{(1)}(0^+) = 0.$$

Proposition 2.1. *For any chosen $\beta \in \mathbb{R}$, the space $\mathscr{S}_\beta(\mathbb{R})$ is a Fréchet space.*

The definition of a Fréchet space can be found, for instance, in [15]. The proof that $\mathscr{S}(\mathbb{R}\backslash\{0\})$ is a Fréchet space follows the same lines of that of [15] for the usual Schwartz space $\mathscr{S}(\mathbb{R})$, and for that reason it will be omitted. Since the spaces $\mathscr{S}_\beta(\mathbb{R})$ are closed vector spaces of $\mathscr{S}(\mathbb{R}\backslash\{0\})$, this implies they are also Fréchet spaces. We notice that along the paper we only use this fact when we invoke the result of [12] about tightness of stochastic process taking values in Fréchet spaces.

Definition 2.6. *We define the operators $\Delta_\beta : \mathscr{S}_\beta(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ and $\nabla_\beta : \mathscr{S}_\beta(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ by*

$$\nabla_\beta H = \begin{cases} H^{(1)}(u), & \text{if } u \neq 0, \\ H^{(1)}(0^+), & \text{if } u = 0, \end{cases} \qquad \text{and} \qquad \Delta_\beta H = \begin{cases} H^{(2)}(u), & \text{if } u \neq 0, \\ H^{(2)}(0^+), & \text{if } u = 0. \end{cases}$$

Notice that the operators ∇_β and Δ_β are essentially the usual derivative and the usual second derivative, but defined in specific domains.

2.3. **Hydrodynamic limit, PDE's and semigroups.** The hydrodynamic limit for the exclusion process with a slow bond was already studied in the papers [4, 5]. We state them here for completeness. Let $g : \mathbb{R} \to [0,1]$ be a continuous by parts function bounded away from zero and one, and let $n \in \mathbb{N}$ be a scaling parameter. We define a probability measure μ^n in Ω by

$$\mu^n\big(\eta(z_1) = 1, \dots, \eta(z_\ell) = 1\big) = \prod_{i=1}^{\ell} g(z_i/n)$$

for any set $\{z_1, \dots, z_\ell\} \subseteq \mathbb{Z}$ and $\ell \in \mathbb{N}$. Let $\{\eta_{tn^2}; t \geq 0\}$ have initial distribution μ^n. We define the *empirical measure* $\{\pi_t^n; t \geq 0\}$ as the measure-valued process given by

$$\pi_t^n(dx) = \frac{1}{n}\sum_{z\in\mathbb{Z}} \eta_{tn^2}(x)\delta_{\frac{x}{n}}(dx).$$

In words, the empirical measure represents the time evolution of the spatial density of particles.

Theorem 2.2 (Franco, Gonçalves, Neumann [4, 5]).
For any $T \geq 0$, as $n \to +\infty$, the sequence of measure valued processes $\{\pi_t^n(dx);\ t \in [0,T]\}_{n\in\mathbb{N}}$ converges in probability with respect to the Skorohod topology of $\mathscr{D}([0,T],\mathscr{M}_+(\mathbb{R}))$, to some $\{u(t,x)dx;\ t \in [0,T]\}$. Moreover,

- *for $\beta \in [0,1)$, $\{u(t,x); t \geq 0, x \in \mathbb{R}\}$ is the unique weak solution of the heat equation*
$$\begin{cases} \partial_t u(t,x) = \partial_{xx}^2 u(t,x), & t \geq 0, x \in \mathbb{R} \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{3}$$
- *for $\beta = 1$, $\{u(t,x); t \geq 0, x \in \mathbb{R}\}$ is the unique weak solution of the heat equation with a boundary condition of Robin's type at $x = 0$*
$$\begin{cases} \partial_t u(t,x) = \partial_{xx}^2 u(t,x), & t \geq 0, x \in \mathbb{R}\backslash\{0\} \\ \partial_x u(t,0^+) = \partial_x u(t,0^-) = \alpha\{u(t,0^+) - u(t,0^-)\}, & t \geq 0 \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{4}$$
- *for $\beta \in (1,+\infty]$, $\{u(t,x); t \geq 0, x \in \mathbb{R}\}$ is the weak solution of the heat equation with a boundary condition of Neumann's type at $x = 0$*
$$\begin{cases} \partial_t u(t,x) = \partial_{xx}^2 u(t,x), & t \geq 0, x \in \mathbb{R}\backslash\{0\} \\ \partial_x u(t,0^+) = \partial_x u(t,0^-) = 0, & t \geq 0 \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases} \tag{5}$$

The papers [4, 5] deal with the finite volume case (periodic). However, the proof for infinite volume is the same, aside from some topological adaptations.

Each one of the partial differential equations mentioned above are linear. As we will see later, in order to prove the existence of a Ornstein-Uhlenbeck process with characteristics Δ_β and ∇_β, we will make use of the explicit expression for the semigroups corresponding to Δ_β. The semigroup of (3) is classical and it acts on $g: \mathbb{R} \to \mathbb{R}$ as

$$T_t g(x) \;=\; \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g(y)\, dy, \quad \text{for } x \in \mathbb{R}. \tag{6}$$

The semigroup of (5) is also known and it is given by

$$T_t^{\text{Neu}} g(x) \;=\; \begin{cases} \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_0^{+\infty} \Big[e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\Big] g(y)\, dy, & \text{for } x > 0, \\ \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_0^{+\infty} \Big[e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\Big] g(-y)\, dy, & \text{for } x < 0. \end{cases} \tag{7}$$

Denote by g_{even} and g_{odd} the even and odd parts of a function $g: \mathbb{R} \to \mathbb{R}$, respectively, or else, for $x \in \mathbb{R}$,

$$g_{\text{even}}(x) = \frac{g(x) + g(-x)}{2} \quad \text{and} \quad g_{\text{odd}}(x) = \frac{g(x) - g(-x)}{2}.$$

Proposition 2.3. *The semigroup of* (4) *acts on $g: \mathbb{R} \to \mathbb{R}$ as*

$$T_t^\alpha g(x) = \frac{1}{\sqrt{4\pi t}} \Bigg\{ \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g_{\text{even}}(y)\, dy$$
$$+ e^{2\alpha x} \int_x^{+\infty} e^{-2\alpha z} \int_0^{+\infty} \Big[\big(\tfrac{z-y+4\alpha t}{2t}\big) e^{-\frac{(z-y)^2}{4t}} + \big(\tfrac{z+y-4\alpha t}{2t}\big) e^{-\frac{(z+y)^2}{4t}} \Big] g_{\text{odd}}(y)\, dy\, dz \Bigg\},$$

for $x>0$, and

$$T_t^\alpha g(x) = \frac{1}{\sqrt{4\pi t}}\Bigg\{ \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} g_{\text{even}}(y)\,dy$$
$$- e^{-2\alpha x}\int_{-x}^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[\big(\tfrac{z-y+4\alpha t}{2t}\big)e^{-\frac{(z-y)^2}{4t}} + \big(\tfrac{z+y-4\alpha t}{2t}\big)e^{-\frac{(z+y)^2}{4t}}\Big]\, g_{\text{odd}}(y)\,dy\,dz\Bigg\}.$$

for $x<0$.

Throughout the text we will simply write T_t^β for the three semigroups T_t, T_t^α and T_t^{Neu}, corresponding to the regimes $\beta\in[0,1)$, $\beta=1$ and $\beta\in(1,+\infty]$, respectively.

Notice that T_t^{Neu} evolves a function in independent ways in each half line, but T_t^α does not. From this characterization of the semigroup T_t^α, we get almost for free the following result:

Proposition 2.4. *Let $u^\alpha:\mathbb{R}_+\times\mathbb{R}\to[0,1]$ be the unique smooth solution of* (4)*, let $u:\mathbb{R}_+\times\mathbb{R}\to[0,1]$ the unique smooth solution of* (3) *and let $u^{Neu}:\mathbb{R}_+\times\mathbb{R}\to[0,1]$ be the unique smooth solution of* (5)*. Then,*

$$\lim_{\alpha\to+\infty} u^\alpha(t,x) \;=\; u(t,x) \quad \textit{and} \quad \lim_{\alpha\to 0} u^\alpha(x,t) \;=\; u^{Neu}(t,x)\,.$$

for all $(t,x)\in\mathbb{R}_+\times(\mathbb{R}\backslash\{0\})$. Besides that, for fixed $t>0$, the following convergence holds

$$\lim_{\alpha\to+\infty}\|u^\alpha(t,\cdot)-u(t,\cdot)\|_{L^p(\mathbb{R})} \;=\; 0 \quad \textit{and} \quad \lim_{\alpha\to 0}\|u^\alpha(t,\cdot)-u^{Neu}(t,\cdot)\|_{L^p(\mathbb{R})} \;=\; 0$$

for all $p\in[1,+\infty]$.

The convergence above can be improved to some extent related to space and time simultaneously. Since this is not the main issue of this paper, we do not enter into details on this.

2.4. **Ornstein-Uhlenbeck process.** Based on [9, 11], we give here a characterization of the generalized Ornstein-Uhlenbeck process which is a solution of

$$d\mathscr{Y}_t = \Delta_\beta \mathscr{Y}_t dt + \sqrt{2\chi(\rho)}\nabla_\beta d\mathscr{W}_t\,, \tag{8}$$

where $\mathscr{W}_t$ is a space-time white noise of unit variance, in terms of a martingale problem. We will see later that this process governs the equilibrium fluctuations of the density of particles. In spite of having a dependence of $\mathscr{Y}_t$ on β, in order to keep notation simple, we do not index on it.

In what follows $\mathscr{S}'_\beta(\mathbb{R})$ denotes the space of bounded linear functionals $f:\mathscr{S}_\beta(\mathbb{R})\to\mathbb{R}$ and $\mathscr{D}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$ (resp. $\mathscr{C}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$) is the space of $\mathscr{S}'_\beta(\mathbb{R})$ càdlàg (resp. continuous) valued functions endowed with the Skohorod topology.

Proposition 2.5. *There exists an unique random element $\mathscr{Y}$ taking values in the space $\mathscr{C}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$ such that:*

i) *For every function $H\in\mathscr{S}_\beta(\mathbb{R})$, $\mathscr{M}_t(H)$ and $\mathscr{N}_t(H)$ given by*

$$\begin{aligned} \mathscr{M}_t(H) &= \mathscr{Y}_t(H)-\mathscr{Y}_0(H)-\int_0^t \mathscr{Y}_s(\Delta_\beta H)ds\,,\\ \mathscr{N}_t(H) &= \big(\mathscr{M}_t(H)\big)^2 - 2\chi(\rho)\,t\,\|\nabla_\beta H\|_{2,\beta}^2 \end{aligned} \tag{9}$$

are $\mathscr{F}_t$-martingales, where for each $t\in[0,T]$, $\mathscr{F}_t:=\sigma(\mathscr{Y}_s(H); s\le t, H\in\mathscr{S}_\beta(\mathbb{R}))$.

ii) $\mathscr{Y}_0$ *is a gaussian field of mean zero and covariance given on* $G,H \in \mathscr{S}_\beta(\mathbb{R})$ *by*

$$\mathbb{E}_\rho^\beta\left[\mathscr{Y}_0(G)\mathscr{Y}_0(H)\right] = \chi(\rho)\int_{\mathbb{R}} G(u)H(u)du\,. \tag{10}$$

Moreover, for each $H \in \mathscr{S}_\beta(\mathbb{R})$*, the stochastic process* $\{\mathscr{Y}_t(H)\,;\, t \geq 0\}$ *is gaussian , being the distribution of* $\mathscr{Y}_t(H)$ *conditionally to* $\mathscr{F}_s$*, for* $s < t$*, normal of mean* $\mathscr{Y}_s(T_{t-s}^\beta)$ *and variance* $\int_0^{t-s}\|\nabla_\beta T_r^\beta H\|_{2,\beta}^2\,dr$*.*

We call the random element $\mathscr{Y}_\cdot$ the generalized Ornstein-Uhlenbeck process of characteristics Δ_β and ∇_β. From the second equation in (9) and Paul Levy's Theorem, the process

$$\mathscr{M}_t(H)(2\chi(\rho)\|\nabla_\beta H\|_{2,\beta}^2)^{-1/2} \tag{11}$$

is a standard brownian motion. Therefore, in view of Proposition 2.5, it makes sense to say that $\mathscr{Y}$ is the formal solution of (8).

2.5. **Equilibrium Density Fluctuations.** In order to establish the Central Limit Theorem for the empirical measure under the invariant state ν_ρ, we need to introduce the density fluctuation field as the linear functional acting on test functions H as:

$$\mathscr{Y}_t^n(H) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} H\left(\frac{x}{n}\right)(\eta_{tn^2}(x)-\rho). \tag{12}$$

We are in position to state the fluctuations for the density of particles.

Theorem 2.6 (C.L.T. for the density of particles)**.**
Consider the Markov process $\{\eta_{tn^2} : t \geq 0\}$ *starting from the invariant state* ν_ρ*. Then, the sequence of processes* $\{\mathscr{Y}_t^n\}_{n\in\mathbb{N}}$ *converges in distribution, as* $n \to +\infty$*, with respect to the Skorohod topology of* $\mathscr{D}([0,T],\mathscr{S}_\beta'(\mathbb{R}))$ *to a gaussian process* $\mathscr{Y}_t$ *in* $\mathscr{C}([0,T],\mathscr{S}_\beta'(\mathbb{R}))$*, which is the formal solution of the Ornstein-Uhlenbeck equation:*

$$d\mathscr{Y}_t = \Delta_\beta \mathscr{Y}_t dt + \sqrt{2\chi(\rho)}\nabla_\beta d\mathscr{W}_t \tag{13}$$

where $\mathscr{W}_t$ *is a space-time white noise of unit variance and the operators* Δ_β *and* ∇_β *were defined in Subsection 2.2.*

2.6. **Equilibrium Current Fluctuations.** Next, we introduce the notion of current of particles through a fixed bond for our microscopic dynamics of generator $\mathscr{L}_n$ evolving on the diffusive time scale tn^2 and starting from the invariant state ν_ρ.

For a site $x \in \mathbb{Z}$, denote by $J_{x,x+1}^n(t)$ the current of particles over the bond $\{x,x+1\}$, which is the total number of jumps from the site x to the site $x+1$ minus the total number of jumps from the site $x+1$ to the site x in the time interval $[0,tn^2]$.

Let $u \in \mathbb{R}$ be a macroscopical point, to which we associate in the microscopical lattice the bond of vertices $\{\lfloor un\rfloor - 1, \lfloor un\rfloor\}$. Here $\lfloor un\rfloor$ denotes the biggest integer smaller than un. To simplify notation, we will simply write

$$J_u^n(t) := J_{\lfloor un\rfloor -1,\lfloor un\rfloor}^n(t)\,.$$

Now, we state the Central Limit Theorem for the current. For that purpose we need to introduce some notation. Denote by $\Phi_{2t}(\cdot)$ the tail of the distribution function of a gaussian random variable with mean zero and variance $2t$, that is, for $x \in \mathbb{R}$,

$$\Phi_{2t}(x) := \int_x^{+\infty}\frac{e^{-u^2/4t}}{\sqrt{4\pi t}}\,du\,.$$

Theorem 2.7 (C.L.T. for the current of particles)**.**
Under $\mathbb{P}^{\beta}_{\rho}$, for every $t \geq 0$ and every $u \in \mathbb{R}$,

$$\frac{J^n_u(t)}{\sqrt{n}} \xrightarrow[n\to+\infty]{} J_u(t)$$

in the sense of finite-dimensional distributions, where $J_u(t)$ is gaussian with covariances given by

- *for $\beta \in [0,1)$,*

$$\mathbb{E}^{\beta}_{\rho}[J_u(t)J_u(s)] = \chi(\rho)\Big(\sqrt{\frac{t}{\pi}}+\sqrt{\frac{s}{\pi}}-\sqrt{\frac{t-s}{\pi}}\Big)\,, \tag{14}$$

that is $J_u(t)$ is a fractional brownian motion of Hurst exponent $1/4$.

- *for $\beta = 1$,*

$$\begin{aligned}\mathbb{E}^{\beta}_{\rho}[J_u(t)J_u(s)] = \chi(\rho)\Big(&\sqrt{\frac{t}{\pi}}+\frac{\Phi_{2t}(2u+4\alpha t)\,e^{4\alpha u+4\alpha^2 t}}{2\alpha}\\ &+\sqrt{\frac{s}{\pi}}+\frac{\Phi_{2t}(2u+4\alpha s)\,e^{4\alpha u+4\alpha^2 s}}{2\alpha}\\ &-\sqrt{\frac{t-s}{\pi}}-\frac{\Phi_{2t}(2u+4\alpha(t-s))\,e^{4\alpha u+4\alpha^2(t-s)}}{2\alpha}-\frac{\Phi_{2t}(2u)}{2\alpha}\Big)\,.\end{aligned}$$

- *for $\beta \in (1,+\infty]$,*

$$\begin{aligned}\mathbb{E}^{\beta}_{\rho}[J_u(t)J_u(s)] = \chi(\rho)\Big(&\sqrt{\frac{t}{\pi}}\Big[1-e^{-u^2/t}\Big]+\sqrt{\frac{s}{\pi}}\Big[1-e^{-u^2/s}\Big]\\ &-\sqrt{\frac{t-s}{\pi}}\Big[1-e^{-u^2/(t-s)}\Big]+2u\,\Phi_{2t}(2u)\Big)\,.\end{aligned} \tag{15}$$

It is of particular interest the covariance at $u=0$, corresponding to the current through the slow bond $\{-1,0\}$. If $\beta \in [0,1)$, the covariance corresponds to the one of a fractional brownian motion of Hurst exponent $1/4$. If $\beta \in (1,+\infty]$, the covariance equals zero as expected, since the Neumann's boundary conditions at $x=0$ make of it an isolated boundary. Finally, for $\beta = 1$, we obtain a family, indexed in the parameter α, of gaussian processes interpolating the fractional brownian motion of parameter $1/4$ and the degenerate process identically equal to zero. Such interpolation is made clear in the next corollary. Before its statement, we emphasize that at the critical value $\beta=1$, the limit of $J^n_u(t)/\sqrt{n}$ does depend on α. Let us denote it by $J^{\alpha}_u(t)$.

Corollary 2.8. *Under $\mathbb{P}^{\beta}_{\rho}$, for every $t \geq 0$ and every $u \in \mathbb{R}$,*

$$J^{\alpha}_u(t) \xrightarrow[\alpha\to+\infty]{} J_u(t)\,,$$

where $J_u(t)$ is the fractional brownian motion with Hurst exponent $1/4$ and

$$J^{\alpha}_u(t) \xrightarrow[\alpha\to 0]{} J_u(t)\,,$$

where $J_u(t)$ is the gaussian process with covariances given by (15)*.*

2.7. **Fluctuations of a tagged particle.** As a consequence of last construction, we are able to deduce the behavior of a single tagged particle as done in [10, 7]. For that purpose, fix $\rho \in (0,1)$, $u > 0$ and consider η_{tn^2} starting from the measure ν_ρ conditioned to have a particle at the site $\lfloor un \rfloor$, that we denote by ν_ρ^u. More precisely, $\nu_\rho^u(\cdot) := \nu_\rho(\,\cdot\,|\eta_{tn^2}(\lfloor un \rfloor) = 1)$. We notice that from symmetry arguments, the same reasoning holds for $u < 0$. We couple the system starting from ν_ρ^u and starting from ν_ρ, in such a way that both processes differ at most in one site at any given time. Then, the analogue of the results stated in Theorems 2.6 and 2.7 for the starting measure ν_ρ^u follow from those results where the system is taken starting from ν_ρ.

Let $X_u(t)$ denote the position at the time tn^2 of a tagged particle initial at the site $\lfloor un \rfloor$. Since we are in dimension one, the order between particles is preserved and as a consequence

$$\{X_u^n(t) \geq n\} = \Big\{J_u^n(t) \geq \sum_{x=\lfloor un \rfloor}^{\lfloor un \rfloor + n - 1} \eta_{tn^2}(x)\Big\}. \tag{16}$$

Last relation together with Theorem 2.7, gives us that

Theorem 2.9 (C.L.T. for a tagged particle)**.**
Under $\mathbb{P}_{\nu_\rho^u}$, every $u \in \mathbb{R}$ and $t \geq 0$

$$\frac{X_u^n(t)}{\sqrt{n}} \xrightarrow[t \to +\infty]{} X_u(t)$$

in the sense of finite-dimensional distributions, where $X_u(t) = J_u(t)/\rho$ in law, where $J_u(t)$ is the same as in Theorem 2.7. In particular, the covariances of the process $X_u(t)$ are given by $\mathbb{E}_\rho^\beta[X_u(t)X_u(s)] = \rho^{-2}\,\mathbb{E}_\rho^\beta[J_u(t)J_u(s)]$.

We do not present the proof of this theorem since it is very similar to the one presented in [7, 10]. We only remark that in this case the mean of the current and the tagged particle is zero since the dynamics is symmetric. For tightness issues we refer the reader to [13], in which the case $\beta = 0$ and $\alpha = 1$ was considered.

We observe that in the case $\beta \in (1, +\infty]$, the tagged particle starting at the origin moves microscopically but we do not see its fluctuations macroscopically, since the variance of $X_0(t)$ equals zero.

3. CENTRAL LIMIT THEOREM OF THE DENSITY OF PARTICLES

In this section we prove Theorem 2.6. As usual in convergence of stochastic process, there are two facts to the be shown: convergence of finite-dimensional distributions of $\mathscr{Y}_t^n$ to those of $\mathscr{Y}_t$ and tightness of the sequence $\{\mathscr{Y}_t^n\}_{n\in\mathbb{N}}$. We start by the former.

3.1. **Characterization of limit points.**

In this section we want to prove that the limit points of the sequence $\{\mathscr{Y}_t^n\}_{n\in\mathbb{N}}$ satisfy Proposition 2.5. We start by showing that any limit point of the sequence $\{\mathscr{Y}_t^n\}_{n\in\mathbb{N}}$ solves (9).

3.1.1. *Martingale problem.* By Dynkin's formula, for a given function $H \in \mathscr{S}_\beta(\mathbb{R})$,

$$\mathscr{M}_t^n(H) = \mathscr{Y}_t^n(H) - \mathscr{Y}_0^n(H) - \int_0^t n^2 \mathscr{L}_n \mathscr{Y}_s^n(H)\,ds \tag{17}$$

is a martingale with respect to the natural filtration $\mathscr{G}_t^n = \sigma(\eta_{sn^2}, s \leq t)$. Doing simple computations we get to

$$\mathscr{M}_t^n(H) = \mathscr{Y}_t^n(H) - \mathscr{Y}_0^n(H) - \mathscr{I}_t^n(H), \tag{18}$$

where

$$\mathscr{I}_t^n(H) = \int_0^t \tfrac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} n^2 \mathbb{L}_n H\big(\tfrac{x}{n}\big)\, \eta_{sn^2}(x)\, ds \tag{19}$$

and $\mathbb{L}_n$ is the generator of the random walk on $\mathbb{Z}$ given on $H:\mathbb{Z}\to\mathbb{R}$ and $x\in\mathbb{Z}$ by:

$$(\mathbb{L}_n H)(\tfrac{x}{n}) = \xi^n_{x,x+1}\left[H(\tfrac{x+1}{n}) - H(\tfrac{x}{n})\right] + \xi^n_{x-1,x}\left[H(\tfrac{x-1}{n}) - H(\tfrac{x}{n})\right].$$

Note that, despite we do not index, the operator $\mathbb{L}_n$ depends on β.

We take in particular $H\in\mathscr{S}_\beta(\mathbb{R})$. By the fact that the sum $\sum_{x\in\mathbb{Z}} n^2\mathbb{L}_n H(\frac{x}{n})$ is null and by adding and subtracting $\int_0^t \mathscr{Y}_s^n(\Delta_\beta H)\,ds$ to $\mathscr{I}_t^n(H)$, we can rewrite the martingale $\mathscr{M}_t^n(H)$ as

$$\mathscr{M}_t^n(H) = \mathscr{Y}_t^n(H) - \mathscr{Y}_0^n(H) - \int_0^t \mathscr{Y}_s^n(\Delta_\beta H)\,ds - R_t^{n,\beta}(H),$$

where

$$R_t^{n,\beta}(H) := \int_0^t \tfrac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{x}{n}\big) - (\Delta_\beta H)(\tfrac{x}{n})\Big\}\, \overline{\eta}_{sn^2}(x)\,ds$$

and for each $x\in\mathbb{Z}$, the centered random variable $\overline{\eta}_{sn^2}(x)$ denotes $\eta_{sn^2}(x)-\rho$.

In some points ahead we will write $\frac{0}{n}$ as zero to emphasize the discretization of space and make easier to follow the computations.

We start by showing that $R_t^{n,\beta}(H)$ is negligible in $L^2(\mathbb{P}_\rho^\beta)$, for all $H\in\mathscr{S}_\beta(\mathbb{R})$.

Proposition 3.1. *For every $t\in[0,T]$, $\beta\in[0,+\infty]$ and $H\in\mathscr{S}_\beta(\mathbb{R})$,*

$$\lim_{n\to+\infty} \mathbb{E}_\rho^\beta\Big[\big(R_t^{n,\beta}(H)\big)^2\Big] = 0.$$

Proof. Separating the sites close to the slow bond, we can rewrite

$$\begin{aligned} R_t^{n,\beta}(H) = & \int_0^t \tfrac{1}{\sqrt{n}} \sum_{x\neq -1,0} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{x}{n}\big) - (\Delta_\beta H)(\tfrac{x}{n})\Big\}\, \overline{\eta}_{sn^2}(x)\,ds \\ & + \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{-1}{n}\big) - (\Delta_\beta H)\big(\tfrac{-1}{n}\big)\Big\}\, \overline{\eta}_{sn^2}(-1)\,ds \\ & + \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{0}{n}\big) - (\Delta_\beta H)\big(\tfrac{0}{n}\big)\Big\}\, \overline{\eta}_{sn^2}(0)\,ds. \end{aligned} \tag{20}$$

The operator Δ_β distinguishes of the usual laplacian operator essentially in the domain. Outside of the macroscopic point 0, for any β, the operator Δ_β behaves as the usual laplacian. Besides that, for $x\neq -1,0$, the term $n^2\mathbb{L}_n(x)$ is exactly the discrete laplacian. Hence, by the classical approximation of the continuous laplacian by the discrete laplacian, the first integral in (20) is $O(1/\sqrt{n})$ and the constant depends only on H.

Since $\Delta_\beta H$ is bounded, in order to show that the sum of the second and third integrals in (20) goes to zero, it is enough to show that

$$r_t^{n,\beta} := \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{-1}{n}\big)\Big\}\, \overline{\eta}_{sn^2}(-1)\,ds + \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ n^2\mathbb{L}_n H\big(\tfrac{0}{n}\big)\Big\}\, \overline{\eta}_{sn^2}(0)\,ds$$

goes to zero as $n\to+\infty$. Recalling the definition of $\mathbb{L}_n$ we arrive at

$$\begin{aligned} r_t^{n,\beta} = & \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ \alpha n^{2-\beta}\big[H\big(\tfrac{0}{n}\big) - H\big(\tfrac{-1}{n}\big)\big] - n^2\big[H\big(\tfrac{-1}{n}\big) - H\big(\tfrac{-2}{n}\big)\big]\Big\}\, \overline{\eta}_{sn^2}(-1)\,ds \\ & + \int_0^t \tfrac{1}{\sqrt{n}} \Big\{ n^2\big[H\big(\tfrac{1}{n}\big) - H\big(\tfrac{0}{n}\big)\big] - \alpha n^{2-\beta}\big[H\big(\tfrac{0}{n}\big) - H\big(\tfrac{-1}{n}\big)\big]\Big\}\, \overline{\eta}_{sn^2}(0)\,ds. \end{aligned} \tag{21}$$

For each regime of β, namely, $\beta \in [0,1)$, $\beta = 1$ and $\beta \in (1,+\infty]$, we present a specific argument to show that $r_t^{n,\beta}$ vanishes as $n \to +\infty$. Let us begin with the

• Case $\beta \in [0,1)$:

Recall that in this case $\mathscr{S}_\beta(\mathbb{R}) = \mathscr{S}(\mathbb{R})$ and thus H is smooth. Let

$$(\Delta_n H)\big(\tfrac{x}{n}\big) = n^2\big[H\big(\tfrac{x+1}{n}\big) + H\big(\tfrac{x-1}{n}\big) - 2H\big(\tfrac{x}{n}\big)\big]$$

be the discrete laplacian. Summing and subtracting suitable increments of H in (21), $r_t^{n,\beta}$ can be rewritten as

$$\int_0^t \tfrac{1}{\sqrt{n}}\Big\{\alpha n^{2-\beta}\big[H(\tfrac{0}{n}) - H(\tfrac{-1}{n})\big] - n^2\big[H(\tfrac{-1}{n}) - H(\tfrac{-2}{n})\big] - (\Delta_n H)(\tfrac{-1}{n})\Big\}\overline{\eta}_{sn^2}(-1)\,ds$$
$$+\int_0^t \tfrac{1}{\sqrt{n}}\Big\{n^2\big[H(\tfrac{1}{n}) - H(\tfrac{0}{n})\big] - \alpha n^{2-\beta}\big[H(\tfrac{0}{n}) - H(\tfrac{-1}{n})\big] - (\Delta_n H)(\tfrac{0}{n})\Big\}\overline{\eta}_{sn^2}(0)\,ds,$$

plus a negligible term in $L^2(\mathbb{P}_\rho^\beta)$, since H is smooth and therefore $\Delta_n H$ is bounded. Then, we have that

$$r_t^{n,\beta} = \int_0^t \tfrac{1}{\sqrt{n}}(\alpha n^{2-\beta} - n^2)\big[H\big(\tfrac{0}{n}\big) - H\big(\tfrac{-1}{n}\big)\big]\big(\overline{\eta}_{sn^2}(-1) - \overline{\eta}_{sn^2}(0)\big)\,ds.$$

Since $n\big[H\big(\tfrac{0}{n}\big) - H\big(\tfrac{-1}{n}\big)\big]$ is bounded, in order to show that $r_t^{n,\beta}$ goes to zero in $L^2(\mathbb{P}_\rho^\beta)$ as $n \to +\infty$, it is enough to show that

$$\lim_{n\to+\infty} \mathbb{E}_\rho^\beta\Big[\Big(\int_0^t \sqrt{n}\big\{\overline{\eta}_{sn^2}(-1) - \overline{\eta}_{sn^2}(0)\big\}\,ds\Big)^2\Big] = 0. \tag{22}$$

For that purpose we will make use of a comparison with empirical averages on boxes of a suitable size. Let

$$\overline{\eta}^\ell(x) \;=\; \frac{1}{\ell}\sum_{y=x}^{x+\ell-1}\overline{\eta}(y), \tag{23}$$

denote the centered empirical average of particles in a box of size ℓ. Summing and subtracting the empirical mean at the sites -1 and 0, and applying the elementary inequality $(a+b+c)^2 \le 4(a^2+b^2+c^2)$, we bound the expectation in (22) from above by:

$$\begin{aligned} &4\,\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t \sqrt{n}\big\{\overline{\eta}_{sn^2}(-1) - \overline{\eta}^\ell_{sn^2}(-1)\big\}\,ds\Big)^2\Big]\\ &+4\,\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t \sqrt{n}\big\{\overline{\eta}^\ell_{sn^2}(-1) - \overline{\eta}^\ell_{sn^2}(0)\big\}\,ds\Big)^2\Big]\\ &+4\,\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t \sqrt{n}\big\{\overline{\eta}^\ell_{sn^2}(0) - \overline{\eta}_{sn^2}(0)\big\}\,ds\Big)^2\Big]. \end{aligned} \tag{24}$$

In order to estimate the first and last expectations, we use Lemma 7.1, which guarantees that they are bounded from above by $Ct(n^{\beta-1} + \ell/n)$. On the other hand, a simple computation shows that the remaining expectation in (24) is bounded from above by $t^2 n/\ell^2$. Putting together the previous computations, we have that

$$\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t \sqrt{n}\big\{\overline{\eta}_{sn^2}(-1) - \overline{\eta}_{sn^2}(0)\big\}\,ds\Big)^2\Big] \le C(tn^{\beta-1} + \frac{t\ell}{n}) + C\frac{t^2 n}{\ell^2}. \tag{25}$$

Choose $\ell := \varepsilon n$. Therefore, letting $n \to +\infty$ and then $\varepsilon \to 0$, the claim (22) follows.

• Case $\beta = 1$:

In this case, by the definition of $\mathscr{S}_\beta(\mathbb{R})$, we have that $\alpha\{H(0^+)-H(0^-)\} = H^{(1)}(0^+) = H^{(1)}(0^-)$. Since $\overline{\eta}_s(-1)$ is bounded,

$$H\big(\tfrac{0}{n}\big)-H\big(\tfrac{-1}{n}\big)=\big[H(0^+)-H(0^-)\big]+O(1/n),$$

and

$$n\big[H\big(\tfrac{-1}{n}\big)-H\big(\tfrac{-2}{n}\big)\big]=H^{(1)}(0^-)+O(1/n),$$

then, it is straightforward to check that the first integral in (21) is of order $O(t/\sqrt{n})$. The same holds for the second integral. Hence, when $\beta=1$, the expression $r_t^{n,\beta}$ is $O(t/\sqrt{n})$, which vanishes as $n\to+\infty$.

• Case $\beta\in(1,+\infty]$:

By definition of $\mathscr{S}_\beta(\mathbb{R})$, since $H^{(1)}(0^+)=H^{(1)}(0^-)=0$, then we can rewrite:

$$r_t^{n,\beta}=\int_0^t n^{3/2-\beta}\big[H\big(\tfrac{0}{n}\big)-H\big(\tfrac{-1}{n}\big)\big]\big(\overline{\eta}_{sn^2}(-1)-\overline{\eta}_{sn^2}(0)\big)\,ds.$$

Since for this range of the parameter β, H is not smooth at the point 0, in order to prove the claim it is enough to show that:

$$\lim_{n\to+\infty}\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t n^{3/2-\beta}\big\{\overline{\eta}_{sn^2}(-1)-\overline{\eta}_{sn^2}(0)\big\}\,ds\Big)^2\Big]=0. \tag{26}$$

By Lemma 7.1 and by summing and subtracting $\eta^\ell_{sn^2}(-1)$ and $\eta^\ell_{sn^2}(0)$ as done above in the case $\beta\in[0,1)$, we can bound the expectation in (26) by $C(tn^{1-\beta}+t\ell n^{1-2\beta}+t^2n^{3-2\beta}/\ell^2)$. Choose $\ell:=\varepsilon n$. Therefore, letting $n\to+\infty$ and then $\varepsilon\to 0$, the claim follows.

□

Now, recall from (18) that $\mathscr{M}_t^n(H)$ is a martingale. In the following section we prove that the sequence $\{\mathscr{Y}_t^n; t\in[0,T]\}_{n\in\mathbb{N}}$ is tight. Moreover, we prove that the sequences $\{\mathscr{I}_t^n; t\in[0,T]\}_{n\in\mathbb{N}}$ and $\{\mathscr{M}_t^n; t\in[0,T]\}_{n\in\mathbb{N}}$ are tight. Assuming last results, let $\{k_n\}n\in\mathbb{N}$ be a subsequence such that all the sequences $\{\mathscr{Y}_t^{k_n}; t\in[0,T]\}_{n\in\mathbb{N}}$, $\{\mathscr{I}_t^{k_n}; t\in[0,T]\}_{n\in\mathbb{N}}$ and $\{\mathscr{M}_t^{k_n}; t\in[0,T]\}_{n\in\mathbb{N}}$ converge. Let $\{\mathscr{Y}_t; t\in[0,T]\}$, $\{\mathscr{I}_t; t\in[0,T]\}$ and $\{\mathscr{M}_t; t\in[0,T]\}$ denote the limit of those sequences.

We want to prove that $\{\mathscr{Y}_t; t\in[0,T]\}_{n\in\mathbb{N}}$ is in $\mathscr{C}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$ and also that for $H\in\mathscr{S}_\beta(\mathbb{R})$:

$$\mathscr{M}_t(H)=\mathscr{Y}_t(H)-\mathscr{Y}_0(H)-\int_0^t\mathscr{Y}_s(\Delta_\beta H)\,ds$$

is a martingale with quadratic variation given by $t\|\nabla_\beta H\|^2_{2,\beta}$. Fix $H\in\mathscr{S}_\beta(\mathbb{R})$. Since we have that for each $n\in\mathbb{N}$, $\mathscr{M}_t^{k_n}(H)$ is a martingale, we want to show that passing to the limit in n we obtain that $\mathscr{M}_t(H)$ is a martingale. By Proposition 4.6 of [8], it is enough to show that $\{\mathscr{M}_t^{k_n}(H)\}_{n\in\mathbb{N}}$ is uniformly integrable. To this end we notice that by Lemma 7.3 we have that

$$\lim_{n\to+\infty}\mathbb{E}_\rho^\beta[(\mathscr{M}_t^{k_n}(H))^2]=2\chi(\rho)t\|\nabla_\beta H\|^2_{2,\beta}.$$

which is enough to conclude.

Now, we prove that the quadratic variation of $\mathscr{M}_t(H)$ is given by $t\|\nabla_\beta H\|^2_{2,\beta}$. Notice that for each $n\in\mathbb{N}$ we have that $(\mathscr{M}_t^{k_n}(H))^2-\langle\mathscr{M}_t^{k_n}(H)\rangle$ is a martingale. Since its quadratic variation converges to $2\chi(\rho)t\|\nabla_\beta H\|^2_{2,\beta}$ we only have to prove that $(\mathscr{M}_t^{k_n}(H))^2$ is

uniformly integrable. For that purpose we prove that $\mathbb{E}^{\beta}_{\rho}[(\mathscr{M}^{n}_{t}(H))^{4}]$ is bounded by a constant that does not depend on n. Now, we can employ, for example, Lemma 3 of [1] which says that there exists a constant C such that

$$\mathbb{E}^{\beta}_{\rho}[(\mathscr{M}^{k_n}_{t}(H))^{4}] \leq C\Big(\mathbb{E}^{\beta}_{\rho}[(\mathscr{M}^{k_n}_{t}(H))^{2}] + \mathbb{E}^{\beta}_{\rho}\Big[\sup_{0\leq t\leq T}\Big|\mathscr{M}^{k_n}_{t}(H) - \mathscr{M}^{k_n}_{t^-}(H)\Big|^{4}\Big]\Big).$$

By Lemma 7.3 the first term on the left hand side of the previous inequality is bounded. On the other hand, since

$$\sup_{0\leq t\leq T}|\mathscr{M}^{n}_{t}(H) - \mathscr{M}^{n}_{t^-}(H)| = \sup_{0\leq t\leq T}|\mathscr{Y}^{n}_{t}(H) - \mathscr{Y}^{n}_{t^-}(H)| \leq \frac{C(H)}{\sqrt{n}},$$

the second term on the right hand side of the previous inequality is also bounded, this finishes the proof.

3.1.2. *Convergence at initial time.*

Proposition 3.2. *$\mathscr{Y}^{n}_{0}$ converges in distribution to $\mathscr{Y}_0$, where $\mathscr{Y}_0$ is a gaussian field with mean zero and covariance given by* (10)*.*

Proof. We claim that, for every $H \in \mathscr{S}_{\beta}(\mathbb{R})$ and every $t > 0$,

$$\lim_{n\to+\infty}\log\mathbb{E}^{\beta}_{\rho}\Big[\exp\{i\theta\mathscr{Y}^{n}_{0}(H)\}\Big] = -\frac{\theta^2}{2}\chi(\rho)\int_{\mathbb{R}}H^{2}(u)du\,. \tag{27}$$

Since ν_ρ is a Bernoulli product measure,

$$\begin{aligned}\log\mathbb{E}^{\beta}_{\rho}[\exp\{i\theta\mathscr{Y}^{n}_{0}(H)\}] &= \log\mathbb{E}^{\beta}_{\rho}\Big[\exp\Big\{\frac{i\theta}{\sqrt{n}}\sum_{x\in\mathbb{Z}}\bar{\eta}_0(x)H\Big(\frac{x}{n}\Big)\Big\}\Big]\\ &= \sum_{x\in\mathbb{Z}}\log\mathbb{E}^{\beta}_{\rho}\Big[\exp\Big\{\frac{i\theta}{\sqrt{n}}\,\bar{\eta}_0(x)H\Big(\frac{x}{n}\Big)\Big\}\Big].\end{aligned}$$

Since H is smooth except possibly at $x = 0$, using Taylor's expansion the right hand side of the last expression is equal to

$$-\frac{\theta^2}{2n}\sum_{x\in\mathbb{Z}}H^2\Big(\frac{x}{n}\Big)\chi(\rho) + O(1/\sqrt{n}).$$

Taking the limit as $n\to+\infty$ and using the continuity of H, the proof of (27) ends. Replacing H by a linear combination of functions and recalling the Crámer-Wold device, the proof ends. $\square$

Remark 3.3. *We notice that the result stated above holds true for $\mathscr{Y}_t$ for any $t \in [0,T]$. In particular we conclude that the gaussian white noise is a stationary solution of* (13)*, for any $\beta \in [0,+\infty]$.*

3.2. **Tightness.** Here we prove tightness of the process $\{\mathscr{Y}^{n}_{t}; t \in [0,T]\}_{n\in\mathbb{N}}$. At first we notice that by the Mitoma's criterium and Proposition 2.1, it is enough to prove tightness of the sequence of real-valued processes $\{\mathscr{Y}^{n}_{t}(H); t \in [0,T]\}_{n\in\mathbb{N}}$, for $H \in \mathscr{S}_{\beta}(\mathbb{R})$.

Proposition 3.4 (Mitoma's criterium [12])**.**
A sequence $\{x_t; t \in [0,T]\}_{n\in\mathbb{N}}$ of processes in $\mathscr{D}([0,T],\mathscr{S}'_{\beta}(\mathbb{R}))$ is tight with respect to the Skorohod topology if and only if the sequence $\{x_t(H); t \in [0,T]\}_{n\in\mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathscr{D}([0,T],\mathbb{R})$, for any $H \in \mathscr{S}_{\beta}(\mathbb{R})$.

Now, to show tightness of the real-valued process we use Aldous' criterium:

Proposition 3.5. *A sequence $\{x_t; t \in [0,T]\}_{n\in\mathbb{N}}$ of real-valued processes is tight with respect to the Skorohod topology of $\mathscr{D}([0,T],\mathbb{R})$ if:*

i) $\displaystyle\lim_{A\to+\infty} \limsup_{n\to+\infty} \mathbb{P}^{\beta}_{\rho}\Big(\sup_{0\le t\le T}|x_t| > A\Big) \;=\; 0\,,$

ii) *for any* $\varepsilon > 0$, $\displaystyle\lim_{\delta\to 0} \limsup_{n\to+\infty} \sup_{\lambda\le\delta} \sup_{\tau\in\mathscr{T}_T} \mathbb{P}^{\beta}_{\rho}(|x_{\tau+\lambda} - x_\tau| > \varepsilon) \;=\; 0\,,$

where $\mathscr{T}_T$ is the set of stopping times bounded by T.

Fix $H \in \mathscr{S}_\beta(\mathbb{R})$. By (18), it is enough to prove tightness of $\{\mathscr{Y}^n_0(H)\}_{n\in\mathbb{N}}$, $\{\mathscr{I}^n_t(H); t\in[0,T]\}_{n\in\mathbb{N}}$, and $\{\mathscr{M}^n_t(H); t\in[0,T]\}_{n\in\mathbb{N}}$. By Proposition 3.2 the sequence of initial fields is obviously tight. For the martingale term, the first claim of the Aldous' criterium is straightforwardly verified as an application of Doob's inequality together with (41). By Lemma 7.6, the first claim can be easily checked for the integral term. It remains to check the second claim, which is more demanding. For that purpose, fix a stopping time $\tau \in \mathscr{T}_T$. By the Chebychev's inequality together with Lemma 7.3 we have that

$$\begin{aligned}
\mathbb{P}^{\beta}_{\rho}\big(\big|\mathscr{M}^n_{\tau+\lambda}(H) - \mathscr{M}^n_{\tau}(H)\big| > \varepsilon\big) &\le \frac{1}{\varepsilon^2}\mathbb{E}^{\beta}_{\rho}\big[\big(\mathscr{M}^n_{\tau+\lambda}(H) - \mathscr{M}^n_{\tau}(H)\big)^2\big]\\
&\le \frac{1}{\varepsilon^2} 2\chi(\rho)\,\lambda\,\|\nabla_\beta H\|^2_{2,\beta}\\
&\le \frac{1}{\varepsilon^2} 2\chi(\rho)\,\delta\,\|\nabla_\beta H\|^2_{2,\beta},
\end{aligned}$$

which vanishes as $\delta \to 0$. In order to check the second claim for the integral term, we use the same argument as above together with Lemma 7.6 to have that

$$\begin{aligned}
\mathbb{P}^{\beta}_{\rho}\big(\big|\mathscr{I}^n_{\tau+\lambda}(H) - \mathscr{I}^n_{\tau}(H)\big| > \varepsilon\big) &\le \frac{1}{\varepsilon^2}\mathbb{E}^{\beta}_{\rho}\big[\big(\mathscr{I}^n_{\tau+\lambda}(H) - \mathscr{I}^n_{\tau}(H)\big)^2\big]\\
&\le \frac{1}{\varepsilon^2}\delta\,\chi(\rho)\|\nabla_\beta H\|^2_{2,\beta},
\end{aligned}$$

which vanishes as $\delta \to 0$. This finishes the proof of tightness.

4. Semigroup results

Here we present the deduction of the explicit formula for the semigroup T^α_t associated to the following heat equation with a boundary condition of robin's type

$$\begin{cases}
\partial_t u(t,x) = \partial^2_{xx} u(t,x), & t\ge 0, x\in\mathbb{R}\backslash\{0\}\\
\partial_x u(t,0^+) = \partial_x u(t,0^-) = \alpha\{u(t,0^+) - u(t,0^-)\}, & t\ge 0\\
u(0,x) = g(x), & x\in\mathbb{R}.
\end{cases} \tag{28}$$

Let T_t be the semigroup associated to the heat equation (3). Let $\tilde{T}^\alpha_t$ be the semigroup related to the following partial differential equation on the half-line:

$$\begin{cases}
\partial_t u(t,x) = \partial^2_{xx} u(t,x), & t\ge 0, x>0\\
\partial_x u(t,0^+) = 2\alpha u(t,0^+), & t\ge 0\\
u(0,x) = g(x), & x>0.
\end{cases} \tag{29}$$

A direct verification shows that

$$T^\alpha_t g(x) \;=\; \begin{cases}
T_t g_{\text{even}}(x) + \tilde{T}^\alpha_t g_{\text{odd}}(x)\,, & \text{for } x>0\,,\\
T_t g_{\text{even}}(x) - \tilde{T}^\alpha_t g_{\text{odd}}(-x)\,, & \text{for } x<0\,,
\end{cases} \tag{30}$$

is solution of (28). Since, the semigroup T_t has the classical expression given in (6), we are therefore left to deduce an explicit expression for $\tilde{T}_t^{\alpha}$. Denote by u the solution of (29) and consider $v = 2\alpha u - \partial_x u$, which is the solution of the following equation

$$\begin{cases} \partial_t v(t,x) = \partial_{xx}^2 v(t,x), & t \geq 0, x > 0 \\ v(t,0^+) = 0, & t \geq 0 \\ v(0,x) = v_0(x), & x > 0. \end{cases}$$

with $v_0(x) = 2\alpha g(x) - \partial_x g(x)$. Last equation is the heat equation with a boundary condition of Dirichlet type. The semigroup $T_t^{\mathrm{Dir}} v_0(x)$ associated to last equation, is classical and is given by

$$T_t^{\mathrm{Dir}} v_0(x) = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} \Big[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \Big] v_0(y)\, dy. \tag{31}$$

Then, we get to

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} \Big[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \Big] \{2\alpha g(x) - \partial_x g(x)\}\, dy.$$

Solving the ordinary linear differential equation $v = 2\alpha u - \partial_x u$ gives that

$$u(t,x) = e^{2\alpha x} \int_x^{+\infty} e^{-2\alpha z} v(t,z)\, dz.$$

From the last two formulas, we arrive at

$$\tilde{T}_t^{\alpha} g(x) = \frac{e^{2\alpha x}}{\sqrt{4\pi t}} \int_x^{+\infty} e^{-2\alpha z} \int_0^{+\infty} \Big[e^{-\frac{(z-x)^2}{4t}} - e^{-\frac{(z+x)^2}{4t}} \Big] \Big(2\alpha g(y) - \partial_y g(y) \Big)\, dy\, dz.$$

Finally, an integration by parts on the term of the integral above involving $\partial_y g$ yields

$$\tilde{T}_t^{\alpha} g(x) = \frac{e^{2\alpha x}}{\sqrt{4\pi t}} \int_x^{+\infty} e^{-2\alpha z} \int_0^{+\infty} \Big[\big(\tfrac{z-y+4\alpha t}{2t}\big) e^{-\frac{(z-y)^2}{4t}} + \big(\tfrac{z+y-4\alpha t}{2t}\big) e^{-\frac{(z+y)^2}{4t}} \Big] g(y)\, dy\, dz. \tag{32}$$

Putting this formula together with (30) and (6) we get the statement of Proposition 2.3.

In possess of the expression of all the semigroups, we can proceed to the

Proof of Proposition 2.4. Recall (30). We claim that

$$\lim_{\alpha \to 0} \tilde{T}_t^{\alpha} g_{\mathrm{odd}}(x) = T_t^{\mathrm{Neu}} g_{\mathrm{odd}}(x) \tag{33}$$

and

$$\lim_{\alpha \to +\infty} \tilde{T}_t^{\alpha} g_{\mathrm{odd}}(x) = T_t^{\mathrm{Dir}} g_{\mathrm{odd}}(x), \tag{34}$$

where T_t^{Dir} is given by (31). We observe that proving (33) and (34) is enough to conclude the proof, since it is of easy verification that

$$T_t g(x) = \begin{cases} T_t g_{\mathrm{even}}(x) + T_t^{\mathrm{Dir}} g_{\mathrm{odd}}(x), & \text{for } x > 0, \\ T_t g_{\mathrm{even}}(x) - T_t^{\mathrm{Dir}} g_{\mathrm{odd}}(-x), & \text{for } x < 0, \end{cases}$$

and

$$T_t^{\mathrm{Neu}} g(x) = \begin{cases} T_t g_{\mathrm{even}}(x) + T_t^{\mathrm{Neu}} g_{\mathrm{odd}}(x), & \text{for } x > 0, \\ T_t g_{\mathrm{even}}(x) - T_t^{\mathrm{Neu}} g_{\mathrm{odd}}(-x), & \text{for } x < 0. \end{cases}$$

Since g_{odd} will have no special role in the convergences (33) and (34), we will write just g instead. We start by showing (33). First, we rewrite (32) to get to

$$\begin{aligned}\tilde{T}_t^{\alpha} g(x) =& \frac{e^{2\alpha x}}{\sqrt{4\pi t}}\int_x^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[(\tfrac{z-y}{2t})e^{-\frac{(z-y)^2}{4t}}+(\tfrac{z+y}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy\,dz\\ &+\frac{2\alpha e^{2\alpha x}}{\sqrt{4\pi t}}\int_x^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\Big[e^{-\frac{(z-y)^2}{4t}}-e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy\,dz\,.\end{aligned}$$

When $\alpha\to 0$, the second parcel on the right hand side of previous equation vanishes. Thus, we are concerned only with the first parcel. Its limit when $\alpha\to 0$ is

$$\frac{1}{\sqrt{4\pi t}}\int_0^{+\infty}\int_x^{+\infty}\Big[(\tfrac{z-y}{2t})e^{-\frac{(z-y)^2}{4t}}+(\tfrac{z+y}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy\,dz\,.$$

Applying Fubini's Theorem to last expression above gives

$$\frac{1}{\sqrt{4\pi t}}\int_0^{+\infty}g(y)\int_x^{+\infty}\Big[(\tfrac{z-y}{2t})e^{-\frac{(z-y)^2}{4t}}+(\tfrac{z+y}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]\,dz\,dy\,.$$

Solving the integral in z, we get that last expression equals to $T_t^{\text{Neu}}g(x)$, as claimed.

Now we prove (34). We begin by splitting (32) as

$$\begin{aligned}\tilde{T}_t^{\alpha} g(x) =& 2\alpha e^{2\alpha x}\int_x^{+\infty} e^{-2\alpha z}\frac{1}{2\alpha}\int_0^{+\infty}\frac{1}{\sqrt{4\pi t}}\Big[(\tfrac{z-y}{2t})e^{-\frac{(z-y)^2}{4t}}+(\tfrac{z+y}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy\,dz\\ &+2\alpha e^{2\alpha x}\int_x^{+\infty} e^{-2\alpha z}\int_0^{+\infty}\frac{1}{\sqrt{4\pi t}}\Big[e^{-\frac{(z-y)^2}{4t}}-e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy\,dz\,.\end{aligned}\tag{35}$$

Since

$$\int_x^{+\infty}e^{-2\alpha z}dz \;=\; \frac{e^{-2\alpha x}}{2\alpha}\,,$$

we can see that the first parcel on right hand side of (35) is an average of the function

$$\frac{1}{2\alpha}\int_0^{+\infty}\frac{1}{\sqrt{4\pi t}}\Big[(\tfrac{z-y}{2t})e^{-\frac{(z-y)^2}{4t}}+(\tfrac{z+y}{2t})e^{-\frac{(z+y)^2}{4t}}\Big]g(y)\,dy \tag{36}$$

over the finite measure $\mathbf{1}_{[x,+\infty)}(z)e^{-2\alpha z}dz$. Since (36) goes to zero when $\alpha\to+\infty$, we are only concerned with the second parcel in (35). By Fubini's Theorem, it is equal to

$$\frac{e^{2\alpha x}}{\sqrt{4\pi t}}\int_0^{+\infty}g(y)\int_x^{+\infty}2\alpha e^{-2\alpha z}\Big[e^{-\frac{(z-y)^2}{4t}}-e^{-\frac{(z+y)^2}{4t}}\Big]\,dz\,dy\,.$$

Performing an integration by parts to the integral in z yields

$$\begin{aligned}\frac{e^{2\alpha x}}{\sqrt{4\pi t}}\int_0^{\infty}g(y)\Bigg[&-e^{-2\alpha z}\Big(e^{-\frac{(z-y)^2}{4t}}-e^{-\frac{(z+y)^2}{4t}}\Big)\Big|_{z=x}^{z=+\infty}\\ &+\int_x^{+\infty}e^{-2\alpha z}\Big(-\frac{(y-z)}{2t}e^{-\frac{(z-y)^2}{4t}}-\frac{(y+z)}{2t}e^{-\frac{(z+y)^2}{4t}}\Big)\,dz\Bigg]dy\end{aligned}$$

which is equal to

$$T_t^{\text{Dir}}g(x)-\frac{e^{2\alpha x}}{\sqrt{4\pi t}}\int_0^{+\infty}g(y)\Bigg[\int_x^{+\infty}e^{-2\alpha z}\Big(\frac{(y-z)}{2t}e^{-\frac{(z-y)^2}{4t}}+\frac{(y+z)}{2t}e^{-\frac{(z+y)^2}{4t}}\Big)\,dz\Bigg]dy\,.$$

Multiplying and dividing the integral term above by 2α, and then applying the same argument on the average previously used, we get that the limit when $\alpha \to +\infty$ is given by $T_t^{\mathrm{Dir}} g(x)$, finishing the proof of the pointwise convergence.

In order to conclude the $L^p(\mathbb{R})$ convergence, we notice that the semigroups are written in terms of the gaussian kernel, from which is not difficult to get a uniform bound in α. Invoking the Dominated Convergence Theorem the proof finishes. □

5. PROOF OF PROPOSITION 2.5

The existence of the Ornstein-Uhlenbeck process solution of (8) was already proved in Section 3. In this section we guarantee that there exists at most one random element $\mathscr{Y}$ taking values in $\mathscr{C}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$ such that *i)* and *ii)* of Proposition 2.5 hold. The next lines follow closely from [11, page 307]. The key result is the equality $T^\beta_{t+\varepsilon}H - T_t^\beta H = \varepsilon\Delta_\beta T_t^\beta H + o(\varepsilon)$, which is well-known for $\beta \in [0,1)$. Since the semigroups T_t^α and T_t^{Neu} are written in terms of the gaussian kernel, the same property holds for them, provided H is in the corresponding domain. In what follows, the same arguments apply for all cases of β, and we will just write T_t^β for the corresponding semigroup.

Fix $H \in \mathscr{S}_\beta(\mathbb{R})$ and $s > 0$. Recall from (11) that $\mathscr{M}_t(H)(2\chi(\rho)\|\nabla_\beta H\|^2_{2,\beta})^{-1/2}$ is a standard Brownian Motion. Therefore, by Itô's Formula, the process $\{X_t^s(H)\,;\, t \geq s\}$ defined by

$$X_t^s(H) = \exp\left\{\frac{1}{2}(t-s)\|\nabla_\beta H\|^2_{2,\beta} + i\Big(\mathscr{Y}_t(H) - \mathscr{Y}_s(H) - \int_s^t \mathscr{Y}_r(\Delta_\beta H)\,dr\Big)\right\}$$

is a martingale. We affirm now that the process $\{Z_t\,,\, 0 \leq t \leq S\}$ defined by

$$Z_t = \exp\left\{\frac{1}{2}\int_0^t \|\nabla_\beta T^\beta_{S-r} H\|^2_{2,\beta}\,dr + i\,\mathscr{Y}_t(T^\beta_{S-t}H)\right\}$$

is also a martingale. To prove this, consider two times $0 \leq t_1 < t_2 \leq S$ and a partition of the interval $[t_1,t_2]$ in n intervals of equal size, or else, $t_1 = s_0 < s_1 < \cdots < s_n = t_2$, with $s_{j+1} - s_j = (t_2 - t_1)/n$. Observe now that

$$\begin{aligned}\prod_{j=0}^{n-1} X^{s_j}_{s_{j+1}}(T^\beta_{S-s_j}H) = \exp\Bigg\{ & \frac{1}{2n}\sum_{j=0}^{n-1}\|\nabla_\beta T^\beta_{S-s_j}H\|^2_{2,\beta} \\ & + i\sum_{j=0}^{n-1}\Big(\mathscr{Y}_{s_{j+1}}(T^\beta_{S-s_j}H) - \mathscr{Y}_{s_j}(T^\beta_{S-s_j}H) - \int_{s_j}^{s_{j+1}} \mathscr{Y}_r(\Delta_\beta T^\beta_{S-s_j}H)\,dr\Big)\Bigg\}.\end{aligned}$$

As $n \to +\infty$, the first sum inside the exponential above converges to

$$\frac{1}{2}\int_{t_1}^{t_2}\|\nabla_\beta T^\beta_{S-r}H\|^2_{2,\beta}\,dr\,.$$

because it is a Riemann sum. The second sum inside the exponential can be rewritten as

$$\mathscr{Y}_{t_2}(T^\beta_{S-t_2+\frac{1}{n}}H) - \mathscr{Y}_{t_1}(T^\beta_{S-t_1}H) + \sum_{j=1}^{n-1}\Big(\mathscr{Y}_{s_j}(T^\beta_{S-s_{j-1}}H - T^\beta_{S-s_j}H) - \int_{s_j}^{s_{j+1}}\mathscr{Y}_r(\Delta_\beta T^\beta_{S-s_j}H)\,dr\Big)\,.$$

Since $\mathscr{Y} \in \mathscr{C}([0,T],\mathscr{S}'_\beta(\mathbb{R}))$, since $T_t^\beta H$ is continuous in time and applying the expansion $T^\beta_{t+\varepsilon}H - T^\beta_t H = \varepsilon\Delta_\beta T^\beta_t H + o(\varepsilon)$, we conclude that the almost sure limit of the the previous

expression is just $\mathscr{Y}_{t_2}(T^{\beta}_{S-t_2}H)-\mathscr{Y}_{t_1}(T^{\beta}_{S-t_1}H)$. Thus, we have obtained that

$$\lim_{n\to+\infty}\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^{\beta}_{S-s_j}H)=\exp\left\{\frac{1}{2}\int_{t_1}^{t_2}\|\nabla_\beta T^{\beta}_{S-r}H\|^2_{2,\beta}\,dr+i\Big(\mathscr{Y}_{t_2}(T^{\beta}_{S-t_2}H)-\mathscr{Y}_{t_1}(T^{\beta}_{S-t_1}H)\Big)\right\},$$

which equals to $\frac{Z_{t_2}}{Z_{t_1}}$ almost surely. Since the complex exponential is bounded, the Dominated Convergence Theorem ensures also the L^1 convergence, which on the other hand implies that

$$\mathbb{E}^{\beta}_{\rho}\Big[G\,\frac{Z_{t_2}}{Z_{t_1}}\Big]=\lim_{n\to+\infty}\mathbb{E}^{\beta}_{\rho}\Big[G\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^{\beta}_{S-s_j}H)\Big],$$

for any bounded function G. Take G bounded and $\mathscr{F}_{t_1}$-measurable. Since for any $H\in\mathscr{S}_\beta(\mathbb{R})$, the process $X^s_t(H)$ is a martingale, by taking the conditional expectation with respect to $\mathscr{F}_{s_{n-1}}$ we can see that

$$\mathbb{E}^{\beta}_{\rho}\Big[G\prod_{j=0}^{n-1}X^{s_j}_{s_{j+1}}(T^{\beta}_{S-s_j}H)\Big]=\mathbb{E}^{\beta}_{\rho}\Big[G\prod_{j=0}^{n-2}X^{s_j}_{s_{j+1}}(T^{\beta}_{S-s_j}H)\Big].$$

By induction, we conclude that

$$\mathbb{E}^{\beta}_{\rho}\Big[G\,\frac{Z_{t_2}}{Z_{t_1}}\Big]=\mathbb{E}^{\beta}_{\rho}\Big[G\Big],$$

for any G bounded and $\mathscr{F}_{t_1}$-measurable, what proves that $\{Z_t\,,\,t\geq 0\}$ is a martingale. From $\mathbb{E}^{\beta}_{\rho}[Z_t|\mathscr{F}_s]=Z_s$, we get

$$\begin{aligned}&\mathbb{E}^{\beta}_{\rho}\Big[\exp\Big\{\frac{1}{2}\int_0^t\|\nabla_\beta T^{\beta}_{S-r}H\|^2_{2,\beta}\,dr+i\,\mathscr{Y}_t(T^{\beta}_{S-t}H)\Big\}\Big|\mathscr{F}_s\Big]\\&=\exp\Big\{\frac{1}{2}\int_0^s\|\nabla_\beta T^{\beta}_{S-r}H\|^2_{2,\beta}\,dr+i\,\mathscr{Y}_s(T^{\beta}_{S-s}H)\Big\},\end{aligned}$$

which in turn gives

$$\mathbb{E}^{\beta}_{\rho}\Big[\exp\Big\{i\,\mathscr{Y}_t(T^{\beta}_{S-t}H)\Big\}\Big|\mathscr{F}_s\Big]=\exp\Big\{-\frac{1}{2}\int_s^t\|\nabla_\beta T^{\beta}_{S-r}H\|^2_{2,\beta}\,dr+i\,\mathscr{Y}_s(T^{\beta}_{S-s}H)\Big\}.$$

Since $T^{\beta}_{S-s}H=T^{\beta}_{t-s}T^{\beta}_{S-t}H$, performing a change of variables in H and then a change of variables in time, we are lead to

$$\mathbb{E}^{\beta}_{\rho}\Big[\exp\Big\{i\,\mathscr{Y}_t(H)\Big\}\Big|\mathscr{F}_s\Big]=\exp\Big\{-\frac{1}{2}\int_0^{t-s}\|\nabla_\beta T^{\beta}_{r}H\|^2_{2,\beta}\,dr+i\,\mathscr{Y}_s(T^{\beta}_{t-s}H)\Big\}.$$

Replacing H by xH, where $x\in\mathbb{R}$, we get that conditionally to $\mathscr{F}_s$, the random variable $\mathscr{Y}_t(H)$ has gaussian distribution of mean $\mathscr{Y}_s(T^{\beta}_{t-s}H)$ and variance $\int_0^{t-s}\|\nabla_\beta T^{\beta}_r H\|^2_{2,\beta}\,dr$. Successive conditioning implies the uniqueness of the finite dimensional distributions of the process $\{\mathscr{Y}_t(H)\,;\,t\in[0,T]\}$, which in turn gives uniqueness in law of the random element $\mathscr{Y}$, finishing the proof.

6. CENTRAL LIMIT THEOREM FOR THE CURRENT

In this section we follow [7, 10, 14]. Recall the definition of the current $J_{x,x+1}(t)$ given in Subsection 2.6. Since the system starts from the equilibrium ν_ρ and the dynamics is symmetric, then $\mathbb{E}^{\beta}_{\rho}[J^n_{x,x+1}(t)]=0$, for any time $t\geq 0$ and any site $x\in\mathbb{Z}$.

For any $x \in \mathbb{Z}$, if the number of particles in the configuration η is finite, we can write

$$J^n_{x,x+1}(t) = \sum_{y \geq x+1} \Big(\eta_{tn^2}(y) - \eta_0(y) \Big).$$

In such case, the current through the bond $\{\lfloor un \rfloor - 1, \lfloor un \rfloor\}$ can be written in terms of the density fluctuation field $\mathscr{Y}^n_t$ as

$$\frac{J^n_u(t)}{\sqrt{n}} = \mathscr{Y}^n_t(H_u) - \mathscr{Y}^n_0(H_u),$$

where H_u is the Heaviside function, or else, $H_u(x) = \mathbf{1}_{[u,+\infty)}(x)$. Our goal is to take the limit as $n \to +\infty$ in the previous equality. At this point we face two problems. Firstly, the equality itself makes no sense unless the configuration η has a finite numbers of particles. Secondly, the Heaviside function does not belong to the space $\mathscr{S}_\beta(\mathbb{R})$. To overcome these difficulties, we notice that by the conservation on the number of particles it holds that

$$J_{x-1,x}(t) - J_{x,x+1}(t) \; = \; \eta_t(x) - \eta_0(x). \tag{37}$$

Next, we define a sequence of functions $G^u_j(x) := (1 - \frac{x-u}{j})^+ H_u(x)$, approximating the Heaviside function H_u. For these functions, the process $\mathscr{Y}^n_t(G^u_j)$ makes sense, no matter the finiteness of the total number of particles. A discrete integration by parts together with (37) gives

$$\mathscr{Y}^n_t(G^u_j) - \mathscr{Y}^n_0(G^u_j) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \Big(G^u_j(\tfrac{x+1}{n}) - G^u_j(\tfrac{x}{n}) \Big) J_{x,x+1}(t)\,.$$

As $j \to +\infty$, the derivative of G^u_j becomes zero except at the discontinuity point $x = u$. This motivates the next lemma:

Lemma 6.1. *For every $t \geq 0$ and for every $\beta \in [0, +\infty]$,*

$$\lim_{j \to +\infty} \mathbb{E}^\beta_\rho \Big[\Big(\frac{J^n_u(t)}{\sqrt{n}} - (\mathscr{Y}^n_t(G^u_j) - \mathscr{Y}^n_0(G^u_j)) \Big)^2 \Big] = 0\,,$$

uniformly over n.

Proof. Recall (18) and (19). A simple computation together with (37) shows that:

$$\frac{J^n_u(t)}{\sqrt{n}} - (\mathscr{Y}^n_t(G^u_j) - \mathscr{Y}^n_0(G^u_j)) = \mathscr{M}^n_t(H_u - G^u_j) + \mathscr{I}^n_t(H_u - G^u_j)$$

By the inequality $(x+y)^2 \leq 2x^2 + 2y^2$, in order to prove the lemma, it is enough to show that the second moment of the two terms on the right hand side of the previous equality vanish as $j \to +\infty$, uniformly over n.

Taking $f = H_u - G^u_j$ in Lemma 7.3 we have that:

$$\mathbb{E}^\beta_\rho[(\mathscr{M}^n_t(f))^2] \leq t \Big\{ 2\chi(\rho) \Big[\tfrac{1}{n} \sum_{x \neq -1} \big(\nabla_n f(\tfrac{x}{n}) \big)^2 + n^{1-\beta} \big(f(\tfrac{0}{n}) - f(\tfrac{-1}{n}) \big)^2 \Big] + O_f(1/j) \Big\}.$$

Hence, by the definition of f we can bound the previous expression by $2\chi(\rho)/j$, which vanishes as $j \to +\infty$. On the other hand, taking $f = H_u - G^u_j$ in Lemma 7.6, we get to

$$\mathbb{E}^\beta_\rho[(\mathscr{I}^n_t(f))^2] \leq 80t \Big\{ \chi(\rho) \Big[\tfrac{1}{n} \sum_{x \neq -1} \big(\nabla_n f(\tfrac{x}{n}) \big)^2 + n^{1-\beta} \big(f(\tfrac{0}{n}) - f(\tfrac{-1}{n}) \big)^2 \Big] + O_f(1/j) \Big\},$$

which can be bounded from above by $80t\chi(\rho)/j$ and vanishes as $j \to +\infty$, finishing the proof of this lemma. ∎

Proof of the Theorem 2.7. The proof follows from the previous lemma and Theorem 2.6. We start with some considerations that work for all $\beta\in[0,+\infty]$.

Fix $j\in\mathbb{N}$ and approximate each G_j^u in $L^2(\mathbb{R})$ by a sequence of smooth functions of compact support, let us say $H_{k,j}^u$. Moreover, choose $H_{k,j}^u$ constant in a neighborhood of zero, which ensures that $H_{k,j}^u\in\mathscr{S}_\beta(\mathbb{R})$. For fixed $t\geq 0$ we have that

$$\mathbb{E}_\rho^\beta\Big[\big(\mathscr{Y}_t^n(H_{k,j}^u)-\mathscr{Y}_t^n(G_j^u)\big)^2\Big]\leq\chi(\rho)\|H_{k,j}^u-G_j^u\|_2^2\,,$$

which vanishes as $k\to+\infty$, by hypothesis. Hence $\mathscr{Y}_t^n(H_{k,j}^u)$ converges to $\mathscr{Y}_t^n(G_j^u)$ in $L^2(\mathbb{P}_\rho^\beta)$, as $k\to+\infty$. By the Theorem 2.6, we have that $\mathscr{Y}_t^n(H_{k,j}^u)$ converges to $\mathscr{Y}_t(H_{k,j}^u)$ in distribution, as $n\to+\infty$. On the other hand, since for all $H,G\in\mathscr{S}_\beta(\mathbb{R})$,

$$\mathbb{E}_\rho^\beta[\mathscr{Y}_t(H)\mathscr{Y}_s(G)]=\chi(\rho)\int_{\mathbb{R}}T_{t-s}^\beta H(v)G(v)dv\,,\tag{38}$$

and since $\mathscr{Y}_t$ is linear, we have that $\mathscr{Y}_t(H_{k,j}^u)$ converges to $\mathscr{Y}_t(G_j^u)$ in L^2, as $k\to+\infty$.

As a consequence, $\mathscr{Y}_t^n(G_j^u)$ converges to $\mathscr{Y}_t(G_j^u)$ in distribution, as $n\to+\infty$. By the previous lemma, $(\mathscr{Y}_t^n(G_j^u)-\mathscr{Y}_0^n(G_j^u))_{j\in\mathbb{N}}$ is a Cauchy sequence uniformly in n. Then, $(\mathscr{Y}_t(G_j^u)-\mathscr{Y}_0(G_j^u))_{j\in\mathbb{N}}$ is a Cauchy sequence and converges, as $j\to+\infty$, to some random variable with gaussian distribution. We denote such limit by $\mathscr{Y}_t(H_u)-\mathscr{Y}_0(H_u)$. Therefore, the normalized current $J_u^n(t)/\sqrt{n}$ converges to a gaussian random variable, which formally reads as $\mathscr{Y}_t(H_u)-\mathscr{Y}_0(H_u)$, where $\mathscr{Y}_t$ is the solution of the Ornstein-Uhlenbeck equation (13). Since the distributions of $\mathscr{Y}_t(H_u)$ are gaussian, this implies the limit current to be gaussian distributed.

The same argument can be applied to show the same result for any vector $(J_u(t_1),..,J_u(t_k))$.

We claim that to compute the covariance, it is enough to compute the variance. Reversibility plus a simple computation together with (38) yields

$$\begin{aligned}\mathbb{E}_\rho^\beta[(J_u(t))^2]=&2\mathbb{E}_\rho^\beta[\mathscr{Y}_0(H_u)(\mathscr{Y}_0(H_u)-\mathscr{Y}_t(H_u))]\\=&2\chi(\rho)\langle H_u,H_u-T_t^\beta H_u\rangle\,.\end{aligned}\tag{39}$$

Above we used (38) despite H_u is not in $\mathscr{S}_\beta(\mathbb{R})$. Nevertheless, by approximating arguments as above one can get that equality for H_u. Then, linearity shows that the covariance can be written as

$$\begin{aligned}\mathbb{E}_\rho^\beta[J_u(t)J_u(s)]&=\chi(\rho)\Big[\langle H_u,H_u-T_t^\beta H_u\rangle+\langle H_u,H_u-T_s^\beta H_u\rangle-\langle H_u,H_u-T_{t-s}^\beta H_u\rangle\Big]\\&=\frac{1}{2}\Big\{\mathbb{E}_\rho^\beta[(J_u(t))^2]+\mathbb{E}_\rho^\beta[(J_u(s))^2]-\mathbb{E}_\rho^\beta[(J_u(t-s))^2]\Big\}\,.\end{aligned}$$

Therefore, we only need to compute the variance for each one of the regimes of β.

• Case $\beta\in[0,1)$.

Recalling (6), we have that

$$\langle H_u,H_u-T_t^\beta H_u\rangle=\int_u^{+\infty}\Big(1-\int_u^{+\infty}\frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4t}dy\Big)dx=\sqrt{\frac{t}{\pi}}\,.$$

From (39) we get

$$\mathbb{E}_\rho^\beta[J_u(t)J_u(s)]=\chi(\rho)\Big(\sqrt{\frac{t}{\pi}}+\sqrt{\frac{s}{\pi}}-\sqrt{\frac{t-s}{\pi}}\Big).$$

• Case $\beta=1$.

Recalling Proposition 2.3, we have that $\langle H_u, H_u - T_t^\beta H_u\rangle$ is equal to

$$\int_u^{+\infty}\Big(1-\int_{-\infty}^{-u}\frac{1}{2\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy-\int_u^{+\infty}\frac{1}{2\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy$$
$$-e^{2\alpha x}\int_x^{+\infty}e^{-2\alpha z}\Big\{\int_u^{+\infty}\frac{z-y+4\alpha t}{2t\sqrt{4\pi t}}e^{-\frac{(z-y)^2}{4t}}dy+\frac{z+y-4\alpha t}{2t\sqrt{4\pi t}}e^{-\frac{(z+y)^2}{4t}}dy\Big\}dz\Big)dx\,,$$

which can be rewritten as

$$\int_u^{+\infty}\Big(\frac{1}{2}+\int_{-u}^{-u}\frac{1}{2\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy$$
$$-e^{2\alpha x}\int_x^{+\infty}\frac{e^{-2\alpha z}}{2}\Big\{-\int_{z-u}^{z+u}\frac{v}{2t\sqrt{4\pi t}}e^{-\frac{v^2}{4t}}dv+2\alpha-2\alpha\Phi_{2t}(z-u)+\Phi_{2t}(z+u)\Big\}dz\Big)dx.$$

A long but elementary computation shows that

$$\langle H_u, H_u - T_t^\beta H_u\rangle=\sqrt{\frac{t}{\pi}}+\frac{2e^{4\alpha u}e^{4\alpha^2 t}\Phi_{2t}(2u+4\alpha t)-2\Phi_{2t}(2u)}{4\alpha}\,,$$

which from (39) is enough to conclude.

• Case $\beta\in(1,+\infty]$.

Recalling (7), we have that

$$\langle H_u, H_u - T_t^\beta H_u\rangle=\int_u^{+\infty}\Big(1-\int_u^{+\infty}\frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4t}dy-\int_u^{+\infty}\frac{1}{\sqrt{4\pi t}}e^{-(x+y)^2/4t}dy\Big)dx$$
$$=\sqrt{\frac{t}{\pi}}\Big[1-e^{-u^2/t}\Big]+2u\Phi_{2t}(2u)\,,$$

which from (39) concludes the proof. □

Proof of Corollary 2.8. In order to prove the result notice that gaussian processes are characterized by its covariance, and the limit of the covariance guarantees the convergence of the processes in the sense of finite dimensional distributions. Thus, it is sufficient to show that

$$\lim_{\alpha\to 0}\frac{e^{4\alpha u+4\alpha^2 t}\Phi_{2t}(2u+4\alpha t)-\Phi_{2t}(2u)}{2\alpha}=2u\Phi_{2t}(2u)-\sqrt{\frac{t}{\pi}}e^{-u^2/t}$$

and

$$\lim_{\alpha\to+\infty}\frac{e^{4\alpha u+4\alpha^2 t}\Phi_{2t}(2u+4\alpha t)-\Phi_{2t}(2u)}{2\alpha}=0.$$

The first limit comes out by L'Hôpital's Rule and the second one is consequence of the estimate $\int_a^{+\infty}e^{-x^2/2}dx\leq\frac{1}{a}e^{-a^2/2}$, for $a\in\mathbb{R}$. □

7. Some useful L^2 estimates

In this section we prove what we call *Local Replacement* which is fundamental in characterizing the limit points of the density fluctuation field.

For a function $g\in L^2(\nu_\rho)$, we denote by $\mathscr{D}_n(g)$ the Dirichlet form of the function g, defined as: $\mathscr{D}_n(g)=\langle g,-\mathscr{L}_n g\rangle_\rho$. An elementary computation shows that

$$\mathscr{D}_n(g)\;=\;\sum_{x\in\mathbb{Z}}\frac{\xi^n_{x,x+1}}{2}\int\Big(g(\eta^{x,x+1})-g(\eta)\Big)^2\,\nu_\rho(d\eta)\,. \tag{40}$$

Lemma 7.1 (*Local Replacement*)**.**
For $\beta\in[0,+\infty]$, any $\ell\geq 1$ and for $x=-1$ it holds that

$$\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t\{\overline{\eta}_{sn^2}(x)-\overline{\eta}^\ell_{sn^2}(x)\}ds\Big)^2\Big]\leq\frac{80t}{n^2}\chi(\rho)\Big(\alpha n^\beta+\ell\Big).$$

In order to prove last lemma, we use the following result:

Lemma 7.2. *For any $g\in L^2(\nu_\rho)$, for a constant $A>0$ and for $x=-1$, it holds that*

$$\int\{\overline{\eta}(x)-\overline{\eta}^\ell(x)\}g(\eta)\nu_\rho(d\eta)\leq A\chi(\rho)(\alpha n^\beta+\ell)+\frac{1}{A}\mathscr{D}_n(g),$$

where $\mathscr{D}_n(g)$ is the Dirichlet form, see (40).

Proof. In order to prove the previous lemma, we notice that by the definition of the empirical average given in (23), we are able to write the integral in the statement of the lemma as

$$\frac{1}{\ell}\sum_{y=x}^{x+\ell-1}\sum_{z=x}^{y-1}\int\{\eta(z)-\eta(z+1)\}g(\eta)\nu_\rho(d\eta).$$

Writing the previous expression as twice its half and performing the change of variables $\eta\mapsto\eta^{z,z+1}$, for which the measure ν_ρ is invariant, we get to

$$\frac{1}{2\ell}\sum_{y=x}^{x+\ell-1}\sum_{z=x}^{y-1}\int(\eta(z)-\eta(z+1))(g(\eta)-g(\eta^{z,z+1}))\nu_\rho(d\eta).$$

Now, by the Cauchy-Schwarz inequality we bound last expression by

$$\begin{aligned}&\frac{1}{2\ell}\sum_{y=x}^{x+\ell-1}\sum_{z=x}^{y-1}\frac{A}{\xi^n_{z,z+1}}\int(\eta(z)-\eta(z+1))^2\nu_\rho(d\eta)\\&+\frac{1}{2\ell}\sum_{y=x}^{x+\ell-1}\sum_{z=x}^{y-1}\frac{\xi^n_{z,z+1}}{A}\int(g(\eta)-g(\eta^{z,z+1}))^2\nu_\rho(d\eta).\end{aligned}$$

To finish the proof it is enough to recall (40).

□

Proof of Lemma 7.1. By Proposition A1.6.1 of [11] we have that

$$\begin{aligned}&\mathbb{E}_\rho^\beta\Big[\Big(\int_0^t\{\overline{\eta}_{sn^2}(x)-\overline{\eta}^\ell_{sn^2}(x)\}ds\Big)^2\Big]\leq 20t\|\overline{\eta}(x)-\overline{\eta}^\ell(x)\|^2_{-1}.\\&=20tC\sup_{g\in L^2(\nu_\rho)}\Big\{2\int\{\overline{\eta}(x)-\overline{\eta}^\ell(x)\}g(\eta)\nu_\rho(d\eta)-n^2\mathscr{D}_n(g)\Big\}\\&\leq 20tC\sup_{g\in L^2(\nu_\rho)}\Big\{2A\chi(\rho)(n^\beta+\ell)+\frac{2}{A}\mathscr{D}_n(g)-n^2\mathscr{D}_n(g)\Big\}.\end{aligned}$$

In last inequality we used the Schwarz inequality together with the previous lemma. Taking $2/A=n^2$ the claim follows. □

Lemma 7.3. *Fix $H\in\mathscr{S}_\beta(\mathbb{R})$. For any $t\geq 0$:*

$$\mathbb{E}_\rho^\beta\big[(\mathscr{M}_t^n(H))^2\big]=t\Big\{2\chi(\rho)\Big[\tfrac{1}{n}\sum_{x\neq -1}\big(\nabla_nH(\tfrac{x}{n})\big)^2+\alpha n^{1-\beta}\big(H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big)^2\Big]+O_H(\tfrac{1}{n})\Big\}\tag{41}$$

and

$$\lim_{n\to+\infty}\mathbb{E}_\rho^\beta[(\mathscr{M}_t^n(H))^2]=2\chi(\rho)t\|\nabla_\beta H\|^2_{2,\beta},$$

where $\mathscr{M}_t^n(H)$ is the martingale defined in (18).

Proof. The quadratic variation of $\mathscr{M}_t^n(H)$ is given by

$$\langle \mathscr{M}^n(H)\rangle_t = \int_0^t n^2\Big[\mathscr{L}_n\mathscr{Y}_s^n(H)^2 - 2\mathscr{Y}_s^n(H)\mathscr{L}_n\mathscr{Y}_s^n(H)\Big]ds$$

A simple computation shows that

$$\begin{aligned}\langle \mathscr{M}^n(H)\rangle_t =& \int_0^t \tfrac{1}{n}\sum_{x\neq -1}(\eta_{sn^2}(x)-\eta_{sn^2}(x+1))^2\Big[n\big(H(\tfrac{x+1}{n})-H(\tfrac{x}{n})\big)\Big]^2 ds\\ &+\int_0^t \alpha n^{1-\beta}(\eta_{sn^2}(-1)-\eta_{sn^2}(0))^2\big(H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big)^2 ds.\end{aligned} \tag{42}$$

To finish the first claim of the lemma is enough to take expectation with respect to ν_ρ in last expression.

Now, we prove the second claim. Since for all $\beta\in[0,+\infty]$, $H\in\mathscr{S}(\mathbb{R}\backslash\{0\})$, the first term on the right side of (41) converges to $2\chi(\rho)t\|\nabla_\beta H\|_2^2$, as $n\to+\infty$. To finish the proof, it is enough to analyze the second term on the right side of (41). For $\beta<1$ since $H\in\mathscr{S}(\mathbb{R})$, then by Taylor's expansion is it easy to check that the second term above is of order $O_H(n^{-\beta})$, which also vanishes as $n\to+\infty$. For $\beta>1$, the second term on the right side of (41) is bounded from above by $tn^{1-\beta}4\|H\|_\infty^2$, which vanishes as $n\to+\infty$. Finally, for $\beta=1$, we use Taylor's expansion and the fact that $\alpha\{H(0^+)-H(0^-)\}=H^{(1)}(0^-)=H^{(1)}(0^+)$ to show that it converges, as $n\to+\infty$, to $2\chi(\rho)t\big(H^{(1)}(0^+)\big)^2$. This concludes the proof. □

Corollary 7.4. *Fix $H\in\mathscr{S}_\beta(\mathbb{R})$ and $t\geq 0$. Then,*

$$|\langle \mathscr{M}^n(H)\rangle_t|\leq t\Big\{\tfrac{1}{n}\sum_{x\neq -1}\big(H^{(1)}(\tfrac{x}{n})\big)^2+n^{1-\beta}\big(H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big)^2+O_H(\tfrac{1}{n})\Big\}. \tag{43}$$

Proof. It is enough to use the triangular inequality in equation (42), together with the fact that $(\eta_{sn^2}(x)-\eta_{sn^2}(x+1))^2\leq 1$, for all $x\in\mathbb{Z}$ and $s\geq 0$. □

Lemma 7.5. *Let g in $L^2(\nu_\rho)$ and $\{F_n\}_n$ a sequence of functions $F_n:\mathbb{R}\to\mathbb{R}$. Then, for all $A>0$,*

$$\int\sum_{x\in\mathbb{Z}}F_n(\tfrac{x}{n})\{\eta(x)-\eta(x+1)\}g(\eta)\nu_\rho(d\eta)\ \leq A\chi(\rho)\sum_{x\in\mathbb{Z}}\frac{1}{\xi^n_{x,x+1}}\big(F_n(\tfrac{x}{n})\big)^2+\tfrac{1}{A}\mathscr{D}_n(g),$$

where $\mathscr{D}_n(g)$ is the Dirichlet form given in (40).

Proof. Rewriting the expression above as twice the half and making the transformation $\eta\mapsto\eta^{z,z+1}$ (for which the probability ν_ρ is invariant), we have

$$\begin{aligned}&\int\sum_{x\in\mathbb{Z}}F_n(\tfrac{x}{n})\{\eta(x)-\eta(x+1)\}g(\eta)\nu_\rho(d\eta)\\ =&\tfrac{1}{2}\int\sum_{x\in\mathbb{Z}}F_n(\tfrac{x}{n})\{\eta(x)-\eta(x+1)\}\{g(\eta)-g(\eta^{x,x+1})\}\nu_\rho(d\eta).\end{aligned}$$

By Cauchy-Schwarz's inequality, for any $A>0$, we bound the previous expression from above by

$$\begin{aligned}
&\tfrac{1}{2}\sum_{x\in\mathbb{Z}}\frac{A}{\xi^n_{x,x+1}}\big(F_n(\tfrac{x}{n})\big)^2\int\{\eta(x)-\eta(x+1)\}^2\,\nu_\rho(d\eta)\\
&+\tfrac{1}{2}\sum_{x\in\mathbb{Z}}\frac{\xi^n_{x,x+1}}{A}\int\{g(\eta)-g(\eta^{x,x+1})\}^2\,\nu_\rho(d\eta).
\end{aligned}$$

This finishes the proof. □

Lemma 7.6. *Fix $H\in\mathscr{S}_\beta(\mathbb{R})$. Then for any $t\geq 0$:*

$$\mathbb{E}^\beta_\rho\Big[\big(\mathscr{I}^n_t(H)\big)^2\Big]\leq 80\,t\,\chi(\rho)\Big\{\tfrac{1}{n}\sum_{x\neq -1}\big(\nabla_nH(\tfrac{x}{n})\big)^2+n^{1-\beta}\big[H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big]^2\Big\},\qquad(44)$$

where $\nabla_nH(\frac{x}{n})=n\big[H(\frac{x+1}{n})-H(\frac{x}{n})\big]$ and

$$\limsup_{n\to+\infty}\mathbb{E}^\beta_\rho\Big[\big(\mathscr{I}^n_t(H)\big)^2\Big]\leq 80\,t\,\chi(\rho)\|\nabla_\beta H\|^2_{2,\beta},$$

where $\mathscr{I}^n_t(H)$ was defined in (19).

Proof. Recall the definition of $\mathscr{I}^n_t(H)$ given in (19). A simple computation shows that

$$\begin{aligned}
\mathscr{I}^n_t(H)=&\int_0^t\sqrt{n}\sum_{x\neq-1,0}\Big\{\nabla_nH(\tfrac{x}{n})-\nabla_nH(\tfrac{x-1}{n})\Big\}\eta_{sn^2}(x)\,ds\\
&+\int_0^t\sqrt{n}\Big\{\nabla_nH(\tfrac{0}{n})\eta_{sn^2}(0)-\nabla_nH(\tfrac{-2}{n})\eta_{sn^2}(-1)\Big\}ds\\
&+\int_0^t n^{3/2-\beta}\big[H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big]\{\eta_{sn^2}(-1)-\eta_{sn^2}(0)\}ds\,.
\end{aligned}$$

Last expression can be rewritten as

$$\int_0^t\sum_{x\in\mathbb{Z}}F_n(\tfrac{x}{n})\{\eta_{sn^2}(x)-\eta_{sn^2}(x+1)\},$$

where

$$F_n(\tfrac{x}{n})=\begin{cases}n^{3/2-\beta}\big[H(\tfrac{0}{n})-H(\tfrac{-1}{n})\big], & \text{if } x=-1,\\[2mm] \sqrt{n}\,\nabla_nH(\tfrac{x}{n}), & \text{otherwise.}\end{cases}$$

By Proposition A1.6.1 of [11], we have that

$$\mathbb{E}^\beta_\rho\Big[\big(\mathscr{I}^n_t(H)\big)^2\Big]\leq 20t\sup_{g\in L^2(\nu_\rho)}\Big\{2\int\sum_{x\in\mathbb{Z}}F_n(\tfrac{x}{n})\{\eta(x)-\eta(x+1)\}g(\eta)\nu_\rho(d\eta)-n^2\mathscr{D}_n(g)\Big\}.$$

By Lemma 7.5, last expression is bounded from above by

$$20t\sup_{g\in L^2(\nu_\rho)}\Big\{2A\chi(\rho)\sum_{x\in\mathbb{Z}}\frac{1}{\xi^n_{x,x+1}}\big(F_n(\tfrac{x}{n})\big)^2+\tfrac{2}{A}\mathscr{D}_n(g)-n^2\mathscr{D}_n(g)\Big\}.$$

Taking $A=\frac{2}{n^2}$ and by the definition of F_n the proof of the first claim ends.

To prove the second one, we notice the following. The first term on the right hand side of (44) converges to $t\,\chi(\rho)\|\nabla_\beta H\|^2_2$. The second term on the right hand side of (44) can be analyzed as in the proof of Lemma 7.3. □

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REFERENCES

[1] Dittrich, P. and Gärtner, J. *A central limit theorem for the weakly asymmetric simple exclusion process*. Math. Nachr., 15, 75–93, (1991).
[2] A. Faggionato, M. Jara, C. Landim: *Hydrodynamic behavior of one dimensional subdiffusive exclusion processes with random conductances*. Probab. Th. and Rel. Fields, 144, no. 3-4, 633-–667, (2009).
[3] Farfan, J ; Simas, A.B. ; Valentim, F. J.: *Equilibrium fluctuations for exclusion processes with conductances in random environments*. Stochastic Process. Appl., 120, no. 8, 1535—1562, (2010).
[4] Franco, T., Gonçalves, P. and Neumann, A.: *Hydrodynamical behavior of symmetric exclusion with slow bonds*, accepted for publication in the Annales de l'Institut Henri Poincaré: Probability and Statistics.
[5] Franco, T., Gonçalves, P. and Neumann, A.: *Phase Transition of a Heat Equation with Robin's Boundary Conditions and Exclusion Process*, arXiv:1210.3662.
[6] T. Franco and C. Landim: *Hydrodynamic Limit of Gradient Exclusion Processes with conductances*. Arch. Ration. Mech. Anal., 195, no. 2, 409—439, (2010).
[7] P. Gonçalves: *Central Limit Theorem for a Tagged Particle in Asymmetric Simple Exclusion*, Stochastic Process and their Applications, 118, 474-502 (2008).
[8] P. Gonçalves and M. Jara: *Universality of KPZ equation*, arXiv:1003.4478.
[9] R. Holley and D. Stroock *Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions*. Publ. Res. Inst. Math. Sci., 14, no. 3, 741–788, (1978).
[10] M. Jara and C. Landim : *Non Equilibrium Central Limit Theorem for a Tagged Particle in Symmetric Simple Exclusion*. Annals Inst. H. Poincaré (B) Probab. and Statist., 42 567–577, (2006).
[11] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 320. Springer-Verlag, Berlin, 1999.
[12] I. Mitoma, *Tightness of probabilities on* $C([0,1];\mathscr{S}')$ *and* $D([0,1];\mathscr{S}')$. Ann. Prob., 11, no. 4, 989–999 (1983).
[13] M. Peligrad and S. Sethuraman, *On fractional Brownian motion limits in one dimensional nearest-neighbor symmetric simple exclusion*, ALEA Lat. Am. J. Probab. Math. Stat., 4, 245—255, (2008).
[14] M. E. Vares and H. Rost *Hydrodynamics of a One-Dimensional Nearest Neighbor Model.* AMS Contemporary Mathematics, 41, 329–342, (1985).
[15] M. Reed and B. Simon. *I: Functional Analysis, Volume 1 (Methods of Modern Mathematical Physics) (vol 1)*. Academic Press, 1 edition, 1981.

UFBA, INSTITUTO DE MATEMÁTICA, CAMPUS DE ONDINA, AV. ADHEMAR DE BARROS, S/N. CEP 40170-110, SALVADOR, BRASIL
E-mail address: tertu@impa.br

CMAT, CENTRO DE MATEMÁTICA DA UNIVERSIDADE DO MINHO, CAMPUS DE GUALTAR, 4710-057, BRAGA, PORTUGAL
E-mail address: patg@math.uminho.pt

UFRGS, INSTITUTO DE MATEMÁTICA, CAMPUS DO VALE, AV. BENTO GONÇALVES, 9500. CEP 91509-900, PORTO ALEGRE, BRASIL
E-mail address: aneumann@impa.br